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Chern-Simons models on $S^2 \times S^1$, torus gauge fixing, and link invariants I

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Abstract

We study Abelian and non-Abelian Chern-Simons models on manifolds M of the form $M = \Sigma \times S^1$, where Σ is a compact oriented surface. By applying the "torus gauge fixing" procedure of Blau and Thompson we derive a heuristic integral formula for the corresponding Wilson loop observables (WLOs) which has some features that make it a promising starting point for the search of a rigorous path integral representation for the WLOs. For the special case $\Sigma = S^2$ and $G = U(1)$, G being the structure group of the model, we indeed obtain a rigorous version of the right-hand side of the aforementioned heuristic formula and thus a rigorous path integral representation of the WLOs in terms of infinite-dimensional oscillatory integrals. This is achieved by combining certain regularization procedures like "loop smearing" and "framing" with methods from white noise analysis. We expect that similar considerations will eventually lead to a rigorous path integral representation of the WLOs also for non-Abelian Chern-Simons models on $M = S^2 \times S^1$ and to a new and purely geometric derivation of the R-matrices of Jones and Turaev.

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1. Introduction

In recent years there has been considerable interest in the Chern-Simons gauge theory. After Witten [32] succeeded in computing the partition function and the Wilson loop observables (WLOs) for various compact base manifolds M and structure groups G the Chern-Simons gauge theory was studied intensively by many different authors, see, e.g., [16,17,8,7,13,5,1,2].

However, even today there are still several important open questions in the field. For example, by now only in the case of Abelian G it has been possible to give a mathematically rigorous realization of the Feynman path integrals representing the Wilson loop observables, see, e.g., [3,25]. The analogous problem for non-Abelian G and general base manifolds M seems to be very hard. If one restricts oneself to base manifolds M of product form $M = \Sigma \times N$, where Σ is an oriented surface and $N \in \{\mathbb{R}, S^1\}$ the situation can be improved by applying suitable gauges which are then available. In [4,20,21] the special case $\Sigma = \mathbb{R}^2$, $N = \mathbb{R}$, was studied in detail and it was shown that using axial gauge fixing it is indeed possible to define and compute the WLOs rigorously. The values for the WLOs which were obtained in [21] are similar to but do not totally agree with those values obtained before in the physics literature. It was conjectured in [21] that the origin for this deviation lies in the fact that the manifold $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ is non-compact. In the present paper we will test this conjecture by studying the compact product manifolds M of the form $M = \Sigma \times S^1$ using a gauge fixing procedure which we will call "quasi-axial gauge fixing" in order to emphasize its similarity to axial gauge fixing in the case of $M = \mathbb{R}^2 \times \mathbb{R}$. Using quasi-axial gauge fixing, which can be applied in those cases where Σ or G is simply-connected, we will first derive a heuristic formula expressing the WLOs by certain multiple integrals, see (6.3). We observe that the two inner integrals in 6.3 are of "Gaussian type" and we therefore expect that by combining constructions from white noise analysis with certain regularization procedures like "loop smearing" and "framing" (cf. Section 9) one can eventually obtain a rigorous version of the right-hand side of (6.3) and consequently also a rigorous definition of the WLOs in terms of path integrals. In fact, for the special case $G = U(1)$ we carry out the details of this approach and later also compute the values of the WLOs explicitly, cf. Sections 7–11.

Before one begins to study this approach more closely also for non-Abelian G it is reasonable to ask first whether one can simplify the integral expressions in (6.3) by replacing "quasi-axial gauge fixing" by a related gauge fixing procedure which was called "torus gauge fixing" in [10,11]. If Σ is non-compact this is indeed possible and as we will show in [18] when carrying out the details of this Ansatz one can obtain rigorous integral representations for the WLOs which can be computed explicitly (cf. Section 11). If Σ is compact, which is the case we are mainly interested in, one has to be more careful because in this case – due to certain topological obstructions (cf. Proposition 3.4 and [12]) – it is not clear whether torus gauge fixing is really a "proper" gauge fixing. Anyhow, we believe that also for compact Σ it will eventually be possible to define and compute the WLOs for non-Abelian G after modifying our approach in a suitable way. In Section 11, we will discuss this point in more detail.

The present paper is organized as follows: after recalling some elementary but important results in Section 2 we introduce in Section 3 quasi-axial and torus gauge fixing for manifolds of the form $\Sigma \times S^1$. We then analyze when quasi-axial gauge (resp. torus gauge) is a "proper" gauge in the sense that at least every "regular" 1-form, i.e., every element of $\mathcal{A}_{\text{reg}}$ (cf. Section 3.1), is gauge equivalent to a quasi-axial 1-form (resp. a 1-form in the torus gauge). In those situations in which quasi-axial gauge and torus gauge are "proper" gauges we then compute the Faddeev–Popov determinants for these gauge fixing procedures, see Section 4. In Section 5, we show how the action function of Chern-Simons models on manifolds of the form $\Sigma \times S^1$ simplifies when restricted to the space of quasi-axial 1-forms.

In Section 6, we combine the results of Sections 4 and 5 and derive the three key formulae of this paper, i.e., Eqs. (6.3), (6.4), and (6.6). In order to find a rigorous realization of the right-hand sides of (6.4) and (6.6) we recall in Section 7 some basic results from white noise analysis. These results are used in Section 8 for finding a rigorous realization of the heuristic integral functional $\int \cdots \mathrm{d}\hat{\mu}^{\perp}_{B}$ appearing in (6.4) and (6.6). If one wants to make rigorous sense of the whole inner integral in (6.4) and (6.6) it seems to be necessary to use two regularization procedures which we call "loop smearing" and "framing". These two regularization procedures are introduced in Section 9.

In Section 10.1, we compute the inner integral in (6.4) for the special case where $G = U(1)$. Finally, in Section 10.2 we also perform the outer integrations appearing in (6.4). In the present paper this is done at a heuristic level. We will sketch in [18] how one can obtain a rigorous treatment of the outer integrations as well.

Note: The first part of the present paper, i.e., Sections 2–5, is based on [19] which was written before we became aware of the work in [10–12]. Although the presentation of the results in Sections 2–5 is tailored to the requirements in Sections 6–11 and is self-contained we recommend to our readers, especially to those with a physics background, also to study the relevant sections in [10–12] as the presentation of the material given there differs considerably from our presentation and provides a complementary point of view.

2. Preliminaries

2.1. Basic definitions

Let M be a connected differentiable manifold and G a compact connected Lie group. Without loss of generality we will assume that G is a Lie subgroup of $U(N)$, $N \in \mathbb{N}$. We will identify the Lie algebra $\mathfrak{g}$ of G with a Lie subalgebra of the Lie algebra $u(N)$ of $U(N)$.

Let Ad denote the right operation of G on itself by inner automorphisms. For every $g \in G$ we will denote the corresponding orbit, i.e., the conjugacy class of g, by $[g]$. The set of all orbits is denoted by $G/Ad(G)$. The vector space of all smooth $\mathfrak{g}$-valued 1-forms on M will be denoted by $\mathcal{A}_M$ or simply by $\mathcal{A}$. By $\mathcal{G}_M$ or by $\mathcal{G}$ we will denote the group of all smooth G-valued mappings on M. It is well-known that the space of connection 1-forms on the trivial principal fiber bundle $P(M, G)$ with group G and base manifold M can be identified with $\mathcal{A}$ and the gauge group of $P(M, G)$ with the group $\mathcal{G}$. Given these identifications the operation of the gauge group on the space of connection 1-forms induces a right-operation $\cdot : \mathcal{A} \times \mathcal{G} \to \mathcal{A}$ given by $A \cdot \Omega := A^{\Omega} := \Omega^{-1}\,\mathrm{d}\Omega + \Omega^{-1} A \Omega$ for $A \in \mathcal{A}$, $\Omega \in \mathcal{G}$. The orbit of an element $A \in \mathcal{A}$ under this operation will be denoted by $[A]$ and the set of all orbits by $\mathcal{A}/\mathcal{G}$.

2.2. Basic results concerning the case $M = S^1$

Let $M = S^1$, let i_{S^1} (or simply i) denote the mapping $[0, 1] \ni u \mapsto \exp(2\pi iu) \in \{z \in \mathbb{C} \mid \|z\| = 1\} \cong S^1$ and set $t_0 := i_{S^1}(0) \in S^1$. The restriction of i_{S^1} onto $[0, 1)$, which is a bijective mapping $[0, 1) \to S^1$, will also be denoted by i_{S^1} and its inverse will be denoted by $i^{-1}_{S^1}$. $i'_{S^1}(u)$, $u \in [0, 1]$, will denote the tangent vector of S^1 in the point $i_{S^1}(u)$ which is induced by the curve i_{S^1}. Finally, by $\partial/\partial t$ we will denote the vector field on S^1 given by $\partial/\partial t(i_{S^1}(u)) = i'_{S^1}(u)$ for $u \in [0, 1]$ and by $\mathrm{d}t$ the real-valued 1-form on S^1 which is dual to $\partial/\partial t$.

For $A \in \mathcal{A}$ let P_A denote the unique solution $P : [0, 1] \to G$ of the ODE $\mathrm{d}/\mathrm{d}t\, P(t) - P(t) \cdot A_{i_{S^1}(t)}(i'_{S^1}(t)) = 0$, $P(0) = 1$. We set $P_t(A) := P_A(t)$, $A \in \mathcal{A}$, $t \in [0, 1]$. Observe that $P_1(A)$ is equal to $\mathrm{Hol}(A; i_{S^1})$, i.e., the holonomy of A around the loop i_{S^1}.

Finally, let $\tilde{\mathcal{G}}$ denote the subgroup of $\mathcal{G}$ given by $\tilde{\mathcal{G}} := \{\Omega \in \mathcal{G} | \Omega(t_0) = 1\}$.

Proposition 2.1.

(i) *The map* $\tilde{j} : \mathcal{A}/\tilde{\mathcal{G}} \to G$ *given by* $\tilde{j}([A]) = \mathrm{Hol}(A; i_{S^1})$ *for all* $A \in \mathcal{A}$ *is a well-defined bijection.*

(ii) *The map* $j : \mathcal{A}/\mathcal{G} \to G/\mathrm{Ad}(G)$ *given by* $j([A]) = [\mathrm{Hol}(A; i_{S^1})]$ for all $A \in \mathcal{A}$ *is a well-defined bijection.*

Proof. We will only prove part (i) of the proposition. The proof of part (ii) will then be obvious. It is easy to see that $\tilde{j}$ is well-defined. Moreover, from the surjectivity of $\exp : \mathfrak{g} \to G$ (G was assumed to be compact and connected) we obtain $\tilde{j}(\mathcal{A}/\tilde{\mathcal{G}}) \supset \{\mathrm{Hol}(B\,\mathrm{d}t; i_{S^1}) \mid B \in \mathfrak{g}\} = \{\exp(B) \mid B \in \mathfrak{g}\} = G$ so $\tilde{j}$ is surjective, too. Finally, let $A, A' \in \mathcal{A}$ such that $\tilde{j}([A]) = \tilde{j}([A'])$, i.e., $\mathrm{Hol}(A; i_{S^1}) = \mathrm{Hol}(A'; i_{S^1})$, and let g denote the mapping $[0, 1] \to G$ given by $g(t) = P_t(A')^{-1} \cdot P_t(A)$ for all $t \in [0, 1]$. From $\mathrm{Hol}(A; i_{S^1}) = \mathrm{Hol}(A'; i_{S^1})$ we get $g(0) = 1 = g(1)$ so the mapping $\tilde{\Omega} := g \circ i^{-1}_{S^1} : S^1 \to G$ is continuous and $\tilde{\Omega}(t_0) = 1$. One can show that $\tilde{\Omega}$ is C^∞, from which $\tilde{\Omega} \in \tilde{\mathcal{G}}$ follows. A short computation then shows that $A = A' \cdot \tilde{\Omega}$. This proves that $\tilde{j}$ is injective. □

Corollary 2.2.

(i) *Let* S *be a subset of* $\mathfrak{g}$ *which fulfills* $\exp(B') \neq \exp(B)$ *for* $B, B' \in S$, $B' \neq B$, *and which has the additional property that* $G = \{\exp(B) \mid B \in S\}$. *Then the set* $\{B\,\mathrm{d}t \mid B \in S\}$ *is a complete and minimal system of representatives of* $\mathcal{A}/\tilde{\mathcal{G}}$.

(ii) *Let* R *be a subset of* $\mathfrak{g}$ *which fulfills* $\exp(B) \neq \exp(B')$ *for* $B, B' \in R$, $B \neq B'$, *and which has the additional property that* $\{\exp(B)|B \in R\}$ *is a complete and minimal system of representatives of* $G/Ad(G)$. *Then the set* $\{B\,\mathrm{d}t \mid B \in R\}$ *is a complete and minimal system of representatives of* $\mathcal{A}/\mathcal{G}$.

Remark 2.1. The corollary above implies that the mapping $\tilde{\Phi}_{S^1} : \mathfrak{g} \times \tilde{\mathcal{G}} \ni (B, \tilde{\Omega}) \mapsto (B\,\mathrm{d}t) \cdot \tilde{\Omega} \in \mathcal{A}$ is surjective. Let T be a maximal torus and let t denote its Lie algebra.

From the fundamental theorem on maximal tori (cf., e.g., Theorem 1.6 in [14], Chap. IV) it follows that also the mapping $\Phi_{S^1} : \mathfrak{t} \times \mathcal{G} \ni (B, \Omega) \mapsto (B\,\mathrm{d}t) \cdot \Omega \in \mathcal{A}$ is surjective: as every element $g \in G$ is contained in a maximal torus and all the maximal tori are pairwise conjugated the set R in the assertion of Corollary 2.2(ii) can always be chosen to be a subset of the Lie algebra $\mathfrak{t}$ of the maximal torus T of G. For example, in the special case where G is semisimple it is possible and natural to choose R such that $\overline{P} \supset R \supset P$, where P is a fixed alcove of $\mathfrak{t}$ (cf. Section V.7 in [14]).

Proposition 2.3. *Let $A \in \mathcal{A}$. The stabilizer S_A of A w.r.t. the $\mathcal{G}$-operation on $\mathcal{A}$ is given by*

$$S_A = \{\omega_g^A \circ i_{S^1}^{-1} \mid g \in \mathcal{C}(\mathrm{Hol}(A; i_{S^1}))\} \subset \mathcal{G}$$

where $\mathcal{C}(\mathrm{Hol}(A; i_{S^1}))$ is the centralizer of $\mathrm{Hol}(A; i_{S^1})$ in G and where $\omega_g^A : [0, 1] \to G$ is given by $\omega_g^A(t) = P_t(A)^{-1} \cdot g \cdot P_t(A)$ for all $t \in [0, 1]$.

Proposition 2.3 is easy to prove if one takes into account that $\mathcal{A}_{S^1}$ (resp. $\mathcal{G}_{S^1}$) can be considered as a subspace of $\mathcal{A}_{\mathbb{R}}$ (resp. as a subgroup of $\mathcal{G}_{\mathbb{R}}$).

From Proposition 2.3 it follows immediately that $\tilde{\mathcal{G}}$ operates freely on $\mathcal{A}$. Combining this with Corollary 2.2(i) we arrive at the following result.

Proposition 2.4. *Let $S \subset \mathfrak{g}$ be as in Corollary 2.2 (i). Then the mapping $\psi_S : G \times \tilde{\mathcal{G}} \to \mathcal{A}$ given by $(\exp(B), \tilde{\Omega}) \mapsto (B\,\mathrm{d}t) \cdot \tilde{\Omega}$ for all $B \in S$, $\tilde{\Omega} \in \tilde{\mathcal{G}}$ is a well-defined bijection.*

3. Quasi-axial gauge fixing and torus gauge fixing for manifolds of the form $M = \Sigma \times S^1$

Let us now assume that M is of the form $M = \Sigma \times S^1$ where Σ is a connected smooth manifold. By $\tilde{\mathcal{G}}$ we denote the subgroup of $\mathcal{G}$ given by $\tilde{\mathcal{G}} := \{\Omega \in \mathcal{G} \mid \Omega((\sigma, t_0)) = 1 \text{ for all } \sigma \in \Sigma\}$.

The 1-form $\mathrm{d}t$ (resp. the vector field $\partial/\partial t$) on S^1 induces a 1-form (resp. a vector field) on M which will again be denoted by $\mathrm{d}t$ (resp. $\partial/\partial t$).

We observe that $\mathcal{A} = \mathcal{A}^{\perp} \oplus \mathcal{A}^{||}$, where

$$\mathcal{A}^{\perp} := \{A \in \mathcal{A} \mid A(\partial/\partial t) = 0\}, \quad \mathcal{A}^{||} := \{B\,\mathrm{d}t \mid B \in C^{\infty}(M, \mathfrak{g})\}$$

For every $A \in \mathcal{A}$, $A^{\perp}$ and $A^{||}$ will denote the unique elements of $\mathcal{A}^{\perp}$ resp. $\mathcal{A}^{||}$ such that $A = A^{\perp} + A^{||}$ holds. Moreover, for a given $A \in \mathcal{A}$ we set $A_0 := A(\partial/\partial t) \in C^{\infty}(M, \mathfrak{g})$, i.e., A_0 is the element of $C^{\infty}(M, \mathfrak{g})$ given by $A^{||} = A_0\,\mathrm{d}t$.

Let T be a maximal torus of G and let us denote the Lie algebra of T by $\mathfrak{t}$. An element A of $\mathcal{A}$ will be called "quasi-axial" (resp. "in the T-torus gauge") if the functions $A_0((\sigma, \cdot))$, $\sigma \in \Sigma$, on S^1 are constant (resp. constant and $\mathfrak{t}$-valued). We will denote the set of all quasi-axial elements (resp. all elements in the T-torus gauge) of $\mathcal{A}$ by $\mathcal{A}^{qax}$ (resp. $\mathcal{A}^{qax}(T)$). Clearly, we have

$$\mathcal{A}^{qax} = \mathcal{A}^{\perp} \oplus \{B\,\mathrm{d}t \mid B \in C^{\infty}(\Sigma, \mathfrak{g})\}, \quad \mathcal{A}^{qax}(T) = \mathcal{A}^{\perp} \oplus \{B\,\mathrm{d}t \mid B \in C^{\infty}(\Sigma, \mathfrak{t})\}$$

(here we have identified $C^{\infty}(\Sigma, \mathfrak{g})$ with the obvious subspace of $C^{\infty}(M, \mathfrak{g})$).

In the following two subsections we will study the two mappings

$$\tilde{\Phi} : \mathcal{A}^{qax} \times \tilde{\mathcal{G}} \ni (A, \tilde{\Omega}) \mapsto A \cdot \tilde{\Omega} \in \mathcal{A}, \quad \Phi : \mathcal{A}^{qax}(T) \times \mathcal{G} \ni (A, \Omega) \mapsto A \cdot \Omega \in \mathcal{A}$$

which generalize the two mappings in Remark 2.1. In particular, we will analyze under what conditions one can expect $\tilde{\Phi}$ and Φ to be "essentially" surjective.

Clearly, if G is Abelian there is a unique maximal torus T of G, namely $T = G$, so in this situation we have $\mathcal{A}^{qax}(T) = \mathcal{A}^{qax}$ and $\tilde{\Phi} = \Phi$.

3.1. *The mapping* $\tilde{\Phi}$

Let G_{reg} denote the set of regular elements of G, i.e., the set of all $g \in G$ which are contained in a unique maximal torus of G. Similarly, let $\mathfrak{g}_{\mathrm{reg}}$ denote the set of regular elements of $\mathfrak{g}$, i.e., the set of all $B \in \mathfrak{g}$ which are contained in a unique maximal Abelian Lie subalgebra of $\mathfrak{g}$. We set $\mathfrak{g}'_{\mathrm{reg}} := \exp^{-1}(G_{\mathrm{reg}})$.

It is not difficult to see that $g \in G_{\mathrm{reg}}$ (resp. $B \in \mathfrak{g}_{\mathrm{reg}}$) if and only if the set of fix points of $\mathrm{Ad}(g)$ (resp. the kernel of $\mathrm{ad}(B)$) is a maximal Abelian Lie subalgebra of $\mathfrak{g}$. From this it follows that $\mathfrak{g}'_{\mathrm{reg}} \subset \mathfrak{g}_{\mathrm{reg}}$.

Let us set $\mathcal{A}_{\mathrm{reg}} := \{A \in \mathcal{A} \mid \forall \sigma \in \Sigma : \mathrm{Hol}(A; (\sigma, i_{S^1})) \in G_{\mathrm{reg}}\}$. Clearly, every $A \in \mathcal{A}_{\mathrm{reg}}$ gives rise to a function $f_A : \Sigma \to G_{\mathrm{reg}}$ given by

$$f_A(\sigma) = \mathrm{Hol}(A; (\sigma, i_{S^1})) = \mathrm{Hol}(A_0((\sigma, \cdot))\,\mathrm{d}t; i_{S^1}) \quad \text{for all } \sigma \in \Sigma \tag{3.1}$$

Remark 3.1. Note that the codimension of $G \backslash G_{\mathrm{reg}}$ in G is at least 3 (cf., e.g., the proof of Lemma 7.5 in [14], Chap. V). This means that if $\dim(\Sigma) \leq 2$ then "almost all" elements of $C^{\infty}(\Sigma, G)$ will be contained in $C^{\infty}(\Sigma, G_{\mathrm{reg}})$ and "almost all" elements of $\mathcal{A}$ will be contained in $\mathcal{A}_{\mathrm{reg}}$. This heuristic argument will be of importance in Section 4.

One can show that the mapping $\exp : \mathfrak{g}'_{\mathrm{reg}} \to G_{\mathrm{reg}}$ is a covering of G_{reg} (cf., e.g., the proof of Prop. 7.11 in V.7 in [14]). If Σ is simply connected the Lifting Theorem implies that every smooth mapping $f : \Sigma \to G_{\mathrm{reg}}$ has a smooth lift w.r.t. this covering. The same is true if G is simply-connected because then also G_{reg} is simply-connected (cf. Sec. V.7 in [14]) and consequently $\exp : \mathfrak{g}'_{\mathrm{reg}} \to G_{\mathrm{reg}}$ is then a trivial covering.

In order to obtain uniqueness for the lift of a smooth mapping $f : \Sigma \to G_{\mathrm{reg}}$ let us fix a point $\sigma_0 \in \Sigma$ and a set S as in Corollary 2.2(i). Then, if Σ or G is simply-connected, there is a unique smooth lift $\tilde{f}$ such that $\tilde{f}(\sigma_0) \in S$ holds. This lift will be denoted by $\tilde{f}_{\sigma_0,S}$.

Proposition 3.1. *If Σ or G is simply-connected and σ_0, S are as above then the mapping*

$$\tilde{\Psi}_{\sigma_0,S} : \mathcal{A}^{\perp} \times C^{\infty}(\Sigma, G_{\mathrm{reg}}) \times \tilde{\mathcal{G}} \ni (A^{\perp}, f, \tilde{\Omega}) \mapsto (A^{\perp} + \tilde{f}_{\sigma_0,S}\mathrm{d}t) \cdot \tilde{\Omega} \in \mathcal{A}$$

is injective and $\mathrm{Image}(\tilde{\Psi}_{\sigma_0,S}) \supset \mathcal{A}_{\mathrm{reg}}$.

Proof.

(i) $\mathrm{Image}(\tilde{\Psi}_{\sigma_0,S}) \supset \mathcal{A}_{\mathrm{reg}}$: Let $A \in \mathcal{A}_{\mathrm{reg}}$ and let $f_A : \Sigma \to G_{\mathrm{reg}}$ be as in (3.1). It is not difficult to see that $f := f_A$ is smooth so $\tilde{f} := \tilde{f}_{\sigma_0,S}$ is well-defined. Now we can apply for every $\sigma \in \Sigma$ the surjectivity statement of Proposition 2.4 and thus we obtain a family $\{\tilde{\Omega}_\sigma \mid \sigma \in \Sigma\}$ of elements of $\tilde{\mathcal{G}}_{S^1}$ such that

$$(\tilde{f}(\sigma)\,\mathrm{d}t) \cdot \tilde{\Omega}_\sigma = A_0(\sigma, \cdot)\,\mathrm{d}t \tag{3.2}$$

where $\mathrm{d}t$ is the 1-form on S^1 defined in Section 2.

One can show that the function $\tilde{\Omega} : M \to G$ given by $\tilde{\Omega}(\sigma, t) = \tilde{\Omega}_\sigma(t)$ is smooth and thus in $\mathcal{G}_M$. From (3.2) and Lemma 1 below it follows $\tilde{f} = (A \cdot \tilde{\Omega}^{-1})_0$. Thus, we get $(A \cdot \tilde{\Omega}^{-1}) - \tilde{f}\,\mathrm{d}t \in \mathcal{A}^\perp$ and finally $A = \tilde{\Psi}_{\sigma_0,S}((A \cdot \tilde{\Omega}^{-1}) - \tilde{f}\,\mathrm{d}t, f, \tilde{\Omega}))$.

(ii) $\tilde{\Psi}_{\sigma_0,S}$ is injective: Let $(A_1^\perp, f_1, \tilde{\Omega}_1), (A_2^\perp, f_2, \tilde{\Omega}_2) \in \mathcal{A}^\perp \times C^\infty(\Sigma, G_{\mathrm{reg}}) \times \tilde{\mathcal{G}}$ with $\tilde{\Psi}_{\sigma_0,S}(A_1^\perp, f_1, \tilde{\Omega}_1) = \tilde{\Psi}_{\sigma_0,S}(A_2^\perp, f_2, \tilde{\Omega}_2) =: A$. Then it is clear that (3.1) holds with f_A replaced by f_1 and with f_A replaced by f_2, from which $f_1 = f_2$ and therefore also $(\tilde{f}_1)_{\sigma_0,S} = (\tilde{f}_2)_{\sigma_0,S} =: \tilde{f}$ follows. So we get $(\tilde{f}(\sigma)\,\mathrm{d}t) \cdot \tilde{\Omega}_1(\sigma, \cdot) = (\tilde{f}_1(\sigma)\,\mathrm{d}t) \cdot \tilde{\Omega}_1(\sigma, \cdot) \overset{(*)}{=} A_0(\sigma, \cdot)\,\mathrm{d}t \overset{(**)}{=} (\tilde{f}_2(\sigma)\,\mathrm{d}t) \cdot \tilde{\Omega}_2(\sigma, \cdot) = (\tilde{f}(\sigma)\,\mathrm{d}t) \cdot \tilde{\Omega}_2(\sigma, \cdot)$ for every $\sigma \in \Sigma$. Here steps $(*)$ and $(**)$ follow from Lemma 1 below. From the injectivity part of Proposition 2.4 it then follows that $\tilde{\Omega}_1(\sigma, \cdot) = \tilde{\Omega}_2(\sigma, \cdot)$ for all $\sigma \in \Sigma$, i.e., $\tilde{\Omega}_1 = \tilde{\Omega}_2$. This implies $A_1^\perp = A_2^\perp$. □

Corollary 3.2. *If Σ or G is simply-connected then* $\mathrm{Image}(\tilde{\Phi}) \supset \mathcal{A}_{\mathrm{reg}}$.

The proof of the following Lemma, which we have used above, is straightforward and will be omitted.

Lemma 1. *Let $\hat{A} \in \mathcal{A}$, $\hat{\Omega} \in \mathcal{G}$. Then $(\hat{A}_0(\sigma, \cdot)\,\mathrm{d}t) \cdot \hat{\Omega}(\sigma, \cdot) = (\hat{A} \cdot \hat{\Omega})_0(\sigma, \cdot)\,\mathrm{d}t$ for all $\sigma \in \Sigma$ where $\mathrm{d}t$ denotes the 1-form on S^1 defined in Section 2.*

Remark 3.2. If G is Abelian then $G = G_{\mathrm{reg}}$ *so if* Σ is simply-connected then $\tilde{\Psi}_{\sigma_0,S} : \mathcal{A}^\perp \times C^\infty(\Sigma, G) \times \tilde{\mathcal{G}} \to \mathcal{A}$ is a bijection and $\tilde{\Phi}$ is surjective.

3.2. The mapping Φ

Let T be a fixed maximal torus of G. The Lie algebra of T will be denoted by $\mathfrak{t}$. Moreover, we set $T_{\mathrm{reg}} := T \cap G_{\mathrm{reg}}$ and $\mathfrak{t}'_{\mathrm{reg}} := \mathfrak{t} \cap \mathfrak{g}'_{\mathrm{reg}}$. Note that $\mathfrak{t}'_{\mathrm{reg}}$ is just the union of the alcoves of $\mathfrak{t}$. One can show that $\exp^{-1}(T_{\mathrm{reg}}) \subset \mathfrak{t}$ which implies $\mathfrak{t}'_{\mathrm{reg}} = \exp^{-1}(T_{\mathrm{reg}})$.

Proposition 3.3. *Let Σ* or *G be simply-connected. Then the following three statements are equivalent*:

(i) $\mathrm{Image}(\Phi) \supset \mathcal{A}_{\mathrm{reg}}$
(ii) $\forall\, f \in C^\infty(\Sigma, G_{\mathrm{reg}}) : \exists g \in C^\infty(\Sigma, G), t \in C^\infty(\Sigma, T_{\mathrm{reg}}) : \quad g \cdot t \cdot g^{-1} = f$
(iii) *Every smooth mapping $h : \Sigma \to G/T$ admits a smooth lift for the fibre bundle $\pi : G \to G/T$.*

Proof. (ii) ⇒ (i): Let $A \in \mathcal{A}_{\mathrm{reg}}$. Set $f := f_A \in C^\infty(\Sigma, G_{\mathrm{reg}})$ and choose $g \in C^\infty(\Sigma, G)$ and $t \in C^\infty(\Sigma, T_{\mathrm{reg}})$ such that $g \cdot t \cdot g^{-1} = f$ holds. Finally, let $\Omega_0 \in \mathcal{G}$ be given by $\Omega_0(\sigma, u) = g(\sigma)$ for all $\sigma \in \Sigma$, $u \in S^1$. From Corollary 3.2 we know that there is a $A^q \in \mathcal{A}^{qax}$ and a $\tilde{\Omega} \in \tilde{\mathcal{G}}$ such that $\tilde{\Phi}(A^q, \tilde{\Omega}) = A$. Clearly, $(A^q \cdot \Omega_0) \cdot (\Omega_0^{-1} \cdot \tilde{\Omega}) = A$ and, taking into account that $\mathfrak{t}'_{\mathrm{reg}} = \exp^{-1}(T_{\mathrm{reg}})$, it is not difficult to see that $A^q \cdot \Omega_0 \in \mathcal{A}^{qax}(T)$ so $A \in \mathrm{Image}(\Phi)$ follows.

(i) $\Rightarrow$ (ii): Let $f \in C^\infty(\Sigma, G_{\mathrm{reg}})$ be arbitrary. As Σ or G is simply connected there is a $B \in C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}})$ such that $\exp \circ B = f$ (cf. the discussion before Proposition 3.1). Set $A := B\,\mathrm{d}t \in \mathcal{A}$. Clearly, $A \in \mathcal{A}_{\mathrm{reg}}$ so from (i) it follows that there is a $A^q \in \mathcal{A}^{qax}(T)$ and a $\Omega \in \mathcal{G}$ such that $\Phi(A^q, \Omega) = A$. Obviously, we have $f_{A^q} \in C^\infty(\Sigma, T_{\mathrm{reg}})$ and $f = f_A = g \cdot f_{A^q} \cdot g^{-1}$, where $g \in C^\infty(\Sigma, G)$ is given by $\Omega^{-1}(\sigma, 0) = g(\sigma)$ for all $\sigma \in \Sigma$.

(iii) $\Rightarrow$ (ii): Let $f \in C^\infty(\Sigma, G_{\mathrm{reg}})$. In order to find a $g \in C^\infty(\Sigma, G)$ and a $t \in C^\infty(\Sigma, T_{\mathrm{reg}})$ with $g \cdot t \cdot g^{-1} = f$ we consider the covering

$$\theta : G/T \times T_{\mathrm{reg}} \ni (gT, t) \mapsto g \cdot t \cdot g^{-1} \in G_{\mathrm{reg}}$$

of G_{reg}, cf. Lemma 7.4 in [14], Chap. V. As Σ is simply-connected (resp. G and therefore also G_{reg} are simply-connected) the Lifting Theorem (resp. the triviality of $\theta : G/T \times T_{\mathrm{reg}} \to G_{\mathrm{reg}}$) implies that f has a lift $(\bar{g}, t) \in C^\infty(\Sigma, G/T) \times C^\infty(\Sigma, T_{\mathrm{reg}}) \cong C^\infty(\Sigma, G/T \times T_{\mathrm{reg}})$. From (iii) it follows that $\bar{g} : \Sigma \to G/T$ can be lifted to a smooth mapping $g : \Sigma \to G$. We see immediately that $g \cdot t \cdot g^{-1} = f$, where t is as above.

(ii) $\Rightarrow$ (iii): Let $h \in C^\infty(\Sigma, G/T)$. For fixed $t_0 \in T_{\mathrm{reg}}$ let $f_{t_0} \in C^\infty(\Sigma, G_{\mathrm{reg}})$ be given by $f_{t_0}(\sigma) = \theta(h(\sigma), t_0)$ for all $\sigma \in \Sigma$. From (ii) it follows that there are functions $g \in C^\infty(\Sigma, G)$ and $t \in C^\infty(\Sigma, T_{\mathrm{reg}})$ such that $g \cdot t \cdot g^{-1} = f_{t_0}$. Setting $\bar{g} := \pi \circ g$, where $\pi : G \to G/T$ is the canonical projection we have $\theta(\bar{g}(\sigma), t(\sigma)) = f_{t_0}(\sigma)$ for all $\sigma \in \Sigma$. The Weyl group $W(G, T) = \mathcal{N}(T)/T$ of (G, T) operates freely from the left on $G/T \times T_{\mathrm{reg}}$ by $nT \cdot (g'T, t') = (g'T \cdot n^{-1}, n \cdot t' \cdot n^{-1}) = (g'n^{-1}T, n \cdot t' \cdot n^{-1})$ for all $n \in \mathcal{N}(T)$, $g' \in G$, $t' \in T$ (note that $n^{-1}T = Tn^{-1}$ if $n \in \mathcal{N}(T)$). The orbits of this operation are just the fibers of the covering $\theta : G/T \times T_{\mathrm{reg}} \to G_{\mathrm{reg}}$ (cf. the proof of Lemma 7.4 in [14], Chap. V). Thus, there is a $n \in \mathcal{N}(T)$ such that $nT \cdot (\bar{g}(\sigma), t(\sigma)) = (h(\sigma), t_0)$ holds for $\sigma = \sigma_0$ and therefore for all $\sigma \in \Sigma$. So $\Sigma \ni \sigma \mapsto g(\sigma) \cdot n^{-1} \in G$ is a smooth lift oft h. □

From Section 4 on we will only study manifolds Σ which are two-dimensional and from Section 5 on we will demand additionally that Σ is oriented. Proposition 3.4 below is adapted to this situation. As we will explain in Remark 3.3 below it follows from the general considerations in [12] that, for G and Σ as in the assumption of Proposition 3.4, statement (iii) and thus also statement (i) of Proposition 3.3 are fulfilled iff Σ is non-compact. For the convenience of the reader we will present an alternative proof of Proposition 3.4 which does not make use of obstruction theory and of universal bundles and is thus (somewhat) more elementary.

Proposition 3.4. *Let G or Σ be simply-connected. Additionally, let us assume that Σ is two-dimensional and oriented and G non-Abelian. Then* $\mathrm{Image}(\Phi) \supset \mathcal{A}_{\mathrm{reg}}$ *holds if and only if Σ is non-compact.*

Proof. First we observe that the assumptions on Σ and G imply that every continuous map $\Sigma \to G$ is 0-homotopic, i.e.

$$\#[\Sigma, G] = 1 \tag{3.3}$$

This can be seen as follows: If G is simply-connected then because of $\pi_2(G) = 0$ (cf., e.g., Theorem 7.1 in [14], Chap. V) it is also 2-connected and as $\dim(\Sigma) = 2$ Eq. (3.3) follows from Cor. 14 in [29], Chap. 7, Sec. 6. If G is not simply-connected then Σ must be simply-connected, which means that we have either $\Sigma \cong \mathbb{R}^2$ or $\Sigma \cong S^2$. If $\Sigma \cong \mathbb{R}^2$ then Eq. (3.3) follows immediately from the fact that $\mathbb{R}^2$ is contractible. If $\Sigma \cong S^2$ then Eq. (3.3) is implied by $\pi_2(G) = 0$.

From Eq. (3.3) and the fact that $\pi : G \to G/T$ is a fibration, i.e., has the homotopy lifting property, we can conclude that statement (iii) of Proposition 3.3 and therefore also $\mathrm{Image}(\Phi) \supset \mathcal{A}_{\mathrm{reg}}$ will hold if and only if

$$\# \left[\Sigma, G/T\right] = 1 \tag{3.4}$$

Let us now distinguish between the following two cases:

(i) *Σ is non-compact*: Let us pick a complex-analytic structure on Σ. Then Σ is a non-compact Riemannian surface and hence a so-called Stein space. From this it follows, that Σ can be embedded bianalytically as a closed subset of $\mathbb{C}^n$ for suitable $n \in \mathbb{N}$. From Theorem 7.2 in [26] it follows that Σ is homotopy equivalent to a one-dimensional CW-complex. As on the other hand G/T is simply-connected (cf. [14], Chap. V, Prop. 7.6) Eq. (3.4) is implied by Cor. 14 in [29], Chap. 7, Sec. 6.

(ii) *Σ is compact*: In this case, Eq. (3.4) does *not* hold. In order to show this it is enough to find two continuous maps $f : \Sigma \to S^2$ and $g : S^2 \to G/T$ such that $(g \circ f)_* : H_2(\Sigma, \mathbb{Z}) \to H_2(G/T, \mathbb{Z})$ is non-trivial. We have assumed above that G is non-Abelian so[1] $\pi_2(G/T) \neq 0$. As $\pi_2(S^2) \cong \mathbb{Z}$ this means that we can find a continuous map $g : S^2 \to G/T$ such that $g_* : \pi_2(S^2) \to \pi_2(G/T)$ is non-trivial. $\pi_2(G/T)$ is torsion-free so g_* is injective. As G/T is simply-connected the Hurewicz Theorem implies that also the induced homomorphism on the second homology groups, i.e., $g_* : H_2(S^2, \mathbb{Z}) \to H_2(G/T, \mathbb{Z})$ is injective. In order to complete the proof of the proposition it is therefore enough to find a continuous map $f : \Sigma \to S^2$ such that $f_* : H_2(\Sigma, \mathbb{Z}) \to H_2(S^2, \mathbb{Z})$ is non-trivial because then $(g \circ f)_* = g_* \circ f_*$ can not be trivial either. But from the assumption that Σ is a closed oriented surface it follows that Σ is either diffeomorphic to S^2 or to the connected sum of finitely many copies of the two-dimensional torus. For all these cases it is easy to find a map f with the desired properties by using, e.g., a suitable triangulation or cell decomposition of Σ. □

Remark 3.3. Another proof of Proposition 3.4 can be obtained as follows: As observed in [12] statement (iii) in Proposition 3.3 is equivalent to the statement that for every smooth map $f : \Sigma \to G/T$ the induced bundle of $\pi : G \to G/T$ (under f) admits a section, i.e., is trivializable. If G is simply-connected then the bundle $\pi : G \to G/T$ is 2-universal (cf. [30]) so every T-bundle on Σ is equivalent to a T-bundle on Σ induced by a map $f : \Sigma \to G/T$. This implies that there is a 1-1-correspondence between the elements of $[\Sigma, G/T]$ and the elements of the set of isomorphy classes of T-bundles on Σ. Using obstruction theory one can show that $\#[\Sigma, G/T] = 1$ iff $H^2(\Sigma, \pi_2(G/T)) = 0$. If Σ is non-compact the fact that

[1] According to [14], Chap. V, Th. 7.1, the group $\pi_2(G/T)$ is isomorphic to the subgroup of the (torsion-free) group $\pi_1(T) \cong Ker(\exp_{|t}) \cong \mathbb{Z}^{\dim(T)}$ which is generated by the inverse roots. For every non-Abelian compact connected Lie group the set of inverse roots is non-empty so $\pi_2(G/T)$ is non-trivial (and torsion-free).

Σ is homotopy equivalent to a one-dimensional CW-complex which we have mentioned above implies $H^2(\Sigma, \pi_2(G/T)) = 0$. If Σ is compact then from Poincaré duality we obtain $H^2(\Sigma, \pi_2(G/T)) \cong H_0(\Sigma, \pi_2(G/T)) \cong \pi_2(G/T) \neq 0$.

4. Heuristic computation of the Faddeev–Popov determinant for quasi-axial gauge fixing and torus gauge-fixing

4.1. The Faddeev–Popov determinant: Some general considerations

Let $M = \Sigma \times S^1$ be as in Section 3. Let N be a $\tilde{\mathcal{G}}$-invariant subset of $\mathcal{A}$ and let $F : \mathcal{A} \to C^\infty(M, \mathfrak{g})$ be a mapping with the property that for all $A \in \mathcal{A}\backslash N$ there is exactly one $\tilde{\Omega}_A \in \tilde{\mathcal{G}}$ such that $F(A^{\tilde{\Omega}_A}) = 0$ holds. Then, using very similar arguments as in [28], one obtains $\int_{\mathcal{A}\backslash N} \chi(A) DA = \int_{\mathcal{A}\backslash N} \chi(A)\Delta[A]\delta(F(A))DA$ for every $\tilde{\mathcal{G}}$-invariant function $\chi : \mathcal{A} \to \mathbb{C}$. Here $\Delta[A]$, $A \in \mathcal{A}\backslash N$, is the "Faddeev–Popov-determinant" given by

$$\Delta[A] = \det \frac{\delta F(A^{\tilde{\Omega}})}{\delta\tilde{\Omega}}_{|\tilde{\Omega}=\tilde{\Omega}_A}. \tag{4.1}$$

If N is sufficiently small that at a heuristic level one would expect

$$\int_{\mathcal{A}} \chi(A) DA = \int_{\mathcal{A}\backslash N} \chi(A) DA \tag{4.2}$$

to hold one finally obtains

$$\int_{\mathcal{A}} \chi(A) DA = \int_{\mathcal{A}\backslash N} \chi(A)\Delta[A]\delta(F(A))DA \tag{4.3}$$

4.2. The Faddeev–Popov determinant for quasi-axial gauge-fixing

Throughout the rest of this paper we will assume that Σ is two-dimensional. Moreover, during the rest of Section 4, we will also demand that Σ or G is simply-connected. Let us fix $\sigma_0 \in \Sigma$ and a set $S \subset \mathfrak{g}$ with the same properties as the set S in Corollary 2.2(i). Let $F^{qax}_{\sigma_0,S} : \mathcal{A} \to C^\infty(M, \mathfrak{g})$ be given by $F^{qax}_{\sigma_0,S}(A) = (\partial/\partial t)A_0 + (1 - 1_S(A_0(\sigma_0, t_0)))$ for $A \in \mathcal{A}$, where A_0 is given by $A^{||} = A_0\, dt$ and where $1_S : \mathfrak{g} \to \{0, 1\}$ is the indicator function of S.

According to Remark 3.1 one can argue at a heuristic level that the $\tilde{\mathcal{G}}$-invariant set $N^{qax} := \mathcal{A}\backslash\mathcal{A}_{\text{reg}}$ is "negligible". Thus, we can expect Eq. (4.2) to hold with N replaced by N^{qax}.

On the other hand it is easy to check that for $A \in \mathcal{A}$ we have $F^{qax}_{\sigma_0,S}(A) = 0$ iff $A \in \mathcal{A}^{qax}$ and simultaneously $A_0(\sigma_0, t_0) \in S$. Thus, Proposition 3.1 implies that for each $A \in \mathcal{A}\backslash N^{qax} = \mathcal{A}_{\text{reg}}$ there is a unique $\tilde{\Omega}_A \in \tilde{\mathcal{G}}$ such that $F^{qax}_{\sigma_0,S}(A^{\tilde{\Omega}_A}) = 0$ holds. Consequently, we obtain Eq. (4.3) with $F = F^{qax}_{\sigma_0,S}$. In order to interpret the informal measure $\delta(F(A))DA$ on $\mathcal{A}\backslash N^{qax} = \mathcal{A}_{\text{reg}}$ note that

$$\{A \in \mathcal{A}_{\text{reg}} \mid F^{qax}_{\sigma_0,S}(A) = 0\} = \mathcal{A}^\perp \oplus C^\infty(\Sigma, \mathfrak{g}'_{\text{reg}}; \sigma_0, S)$$

where $C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}}; \sigma_0, S) := \{B \mid B \in C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}}), B(\sigma_0) \in S\}$. So we make the Ansatz $\delta(F(A))DA = DA^\perp \otimes D_S B$ where $DA^\perp$ is the informal "Lebesgue measure" on $\mathcal{A}^\perp$, and where $D_S B$ is the measure on $C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}}; \sigma_0, S)$ obtained as the image of $Dg_{|C^\infty(\Sigma, G_{\mathrm{reg}})}$ under the mapping $C^\infty(\Sigma, G_{\mathrm{reg}}) \ni f \mapsto \tilde{f}_{\sigma_0, S} \in C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}}; \sigma_0, S)$. Here Dg is the informal Haar measure on $C^\infty(\Sigma, G)$.

From Eq. (4.1) with $F = F^{qax}_{\sigma_0, S}$ we get for $A^\perp \in \mathcal{A}^\perp$, $B \in C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}}; \sigma_0, S)$ (taking into account that $\tilde{\Omega}_A \equiv 1$ for $A := A^\perp + B\,\mathrm{d}t$)

$$\Delta[A^\perp + B\,\mathrm{d}t] \sim |\det((\partial/\partial t + \mathrm{ad}(B)) \cdot \partial/\partial t)| \sim |\det(\partial/\partial t + \mathrm{ad}(B))| =: \tilde{\Delta}[B] \tag{4.4}$$

where with $\partial/\partial t + \mathrm{ad}(B)$ and $\partial/\partial t$ we mean the obvious operators on $C^\infty_{\mathfrak{g}}(\Sigma \times S^1)$ and where $\sim$ denotes equality up to a constant independent of B. So Eq. (4.3) now reads

$$\int_{\mathcal{A}} \chi(A)\mathrm{D}A \sim \int_{C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}}; \sigma_0, S)} \left[\int_{\mathcal{A}^\perp} \chi(A^\perp + B\,\mathrm{d}t) DA^\perp\right] \tilde{\Delta}[B] D_S B \tag{4.5}$$

with $\sim$ denoting equality up to a constant independent of χ.

One can show that there is a sequence $(S_i)_{i \in \mathbb{N}}$ of subsets of $\mathfrak{g}$ with the same properties as the set S above such that $\mathfrak{g}'_{\mathrm{reg}} = \coprod_{i=1}^\infty (S_i \cap \mathfrak{g}'_{\mathrm{reg}})$ holds where $\coprod$ denotes "disjoint union". For such a sequence $(S_i)_{i \in \mathbb{N}}$ we have

$$C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}}) = \coprod_{i=1}^{\infty} C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}}; \sigma_0, S_i)$$

For each S_i we can derive an analogue of Eq. (4.5) and by "averaging" over the right-hand sides of these analogues of (4.5) we obtain

$$\int_{\mathcal{A}} \chi(A)\mathrm{D}A \sim \int_{C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}})} \left[\int_{\mathcal{A}^\perp} \chi(A^\perp + B\,\mathrm{d}t) DA^\perp\right] \tilde{\Delta}[B] DB \tag{4.6}$$

where $DB := \sum_i D_{S_i} B$. Here we have identified each $D_{S_i} B$ with the obvious measure on $C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}})$.

Heuristically, one should expect that DB is "of product form", i.e. $DB = (\otimes^\Sigma \mu_{\mathfrak{g}'_{\mathrm{reg}}})_{|C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}})}$ where $\mu_{\mathfrak{g}'_{\mathrm{reg}}}$ is a suitable measure on $\mathfrak{g}'_{\mathrm{reg}}$. More precisely, we should have $\mu_{\mathfrak{g}'_{\mathrm{reg}}} = (\exp_{|\mathfrak{g}'_{\mathrm{reg}}})^* \mu_G$ where $(\exp_{|\mathfrak{g}'_{\mathrm{reg}}})^* \mu_G$ denotes the "pullback" of the normalized Haar measure μ_G on G w.r.t. $\exp_{|\mathfrak{g}'_{\mathrm{reg}}}$ (the notion of pullback defined with the help of the associated volume forms w.r.t. to fixed orientations on G and $\mathfrak{g}$). By taking into account that the differential of exp at the point $B_0 \in \mathfrak{g}'_{\mathrm{reg}}$ is given by $(\mathrm{d}\exp(B_0)) = \exp(B_0) \cdot \sum_{n=0}^\infty (\mathrm{ad}(B_0))^n/(n+1)!$ and by using similar arguments[2] as in the proof of Prop. 1.8 in Chap. IV in [14] we obtain $\mu_{\mathfrak{g}'_{\mathrm{reg}}}(\mathrm{d}B_0) \sim \det(\sum_{n=0}^\infty (\mathrm{ad}(B_0))^n/(n+1)!)\lambda_{\mathfrak{g}}(\mathrm{d}B_0)$

[2] Note that our ad differs from the ad in [14] by a minus sign

where $\lambda_{\mathfrak{g}}$ is the Lebesgue measure on $\mathfrak{g}$. This suggests the heuristic formula

$$DB = \det\left(\sum_{n=0}^{\infty}\frac{(\mathrm{ad}(B))^n}{(n+1)!}\right)\mathrm{d}B \tag{4.7}$$

where $\mathrm{d}B$ is the informal Lebesgue measure on $C^{\infty}(\Sigma,\mathfrak{g})$. In particular, for Abelian G we should have $DB = \mathrm{d}B$. In Section 4.3 below, the following variant of (4.6) will be useful:

$$\int_{\mathcal{A}}\chi(A)DA \sim \int_{C^{\infty}(\Sigma,S^*)}\left[\int_{\mathcal{A}^{\perp}}\chi(A^{\perp}+B\,\mathrm{d}t)DA^{\perp}\right]\tilde{\Delta}[B]D_{S^*}B \tag{4.8}$$

Here S^* is a fixed connected component of $\mathfrak{g}'_{\mathrm{reg}}$ and $D_{S^*}B := \sum_{i\in I} D_{S_i}B$ where $(S_i)_{i\in I}$ is a sequence of subsets of $\mathfrak{g}$ with the same properties as the set S above such that $S^* = \coprod_{i\in I}(S_i\cap\mathfrak{g}'_{\mathrm{reg}})$ holds (note that I can be finite, e.g., in the case where G is semisimple).

Remark 4.1. For functions $\chi:\mathcal{A}\to\mathbb{C}$ with the property that $\int_{\mathcal{A}^{\perp}}\chi(A^{\perp}+B\,\mathrm{d}t)\Delta[A^{\perp}+B\,\mathrm{d}t]DA^{\perp}$ is of the form $f(\exp(B))$ for a suitable function f one obtains immediately from Eq. (4.5) (using $\exp(\tilde{f}_{\sigma_0,S}) = f$) $\int_{\mathcal{A}}\chi(A)DA \sim \int f(\exp(B))D_SB = \int_{C^{\infty}(\Sigma,G_{\mathrm{reg}})}f(g)Dg$ where Dg is as above. If f is even a cylindrical function one can replace the space $C^{\infty}(\Sigma,G_{\mathrm{reg}})$ by the space $(G_{\mathrm{reg}})^{\Sigma}$ of arbitrary G_{reg}-valued functions on Σ and the informal measure Dg by the product measure $\otimes^{\Sigma}\left((\mu_G)_{|G_{\mathrm{reg}}}\right)$, where μ_G is the normalized Haar measure on G. Note that, even though Σ is uncountable, the measure $\otimes^{\Sigma}\left((\mu_G)_{|G_{\mathrm{reg}}}\right)$ is mathematically well-defined, cf., e.g., [9].

4.3. *An analogue of (4.6) for torus gauge-fixing*

Let $(\cdot,\cdot)_{\mathfrak{g}}$ denote the (well-defined) scalar product $\mathfrak{g}\times\mathfrak{g}\ni(A,B)\mapsto -\mathrm{Tr}(AB)\in\mathbb{R}$ (!) on $\mathfrak{g}$. The norm associated to $(\cdot,\cdot)_{\mathfrak{g}}$ will be denoted by $|\cdot|_{\mathfrak{g}}$. We fix once and for all a maximal torus T of G. By $\mathfrak{g}_0$ we denote the $(\cdot,\cdot)_{\mathfrak{g}}$-orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$, where $\mathfrak{t}$ is the Lie algebra of T.

Let us consider the right-operation of the group $\mathcal{G}_{\Sigma} = C^{\infty}(\Sigma,G)$ on the space $C^{\infty}(\Sigma,\mathfrak{g})$ given by $B\cdot\Omega_0 = \Omega_0^{-1}B\Omega_0$ for $B\in C^{\infty}(\Sigma,\mathfrak{g})$, $\Omega_0\in\mathcal{G}_{\Sigma}$. If $\mathcal{G}_{\Sigma}$ is identified with the obvious subgroup of $\mathcal{G}=\mathcal{G}_M = C^{\infty}(M,G)$ then $\mathcal{G}_{\Sigma}$ also operates on $\mathcal{A}=\mathcal{A}_M$. As this operation is linear and as $\mathcal{G}_{\Sigma}$ leaves the subspace $\mathcal{A}^{\perp}$ of $\mathcal{A}$ invariant we have for every $\mathcal{G}$-invariant function χ on $\mathcal{A}$, every $\Omega_0\in\mathcal{G}_{\Sigma}\subset\mathcal{G}$ and every $B\in C^{\infty}(\Sigma,\mathfrak{g})$:

$$\begin{aligned}\int\chi(A^{\perp}+B\,\mathrm{d}t)DA^{\perp} &= \int\chi((A^{\perp}+B\,\mathrm{d}t)\cdot\Omega_0)DA^{\perp}\\ &= \int\chi(A^{\perp}\cdot\Omega_0+(\Omega_0^{-1}B\Omega_0)\,\mathrm{d}t)DA^{\perp}\\ &= \int\chi(A^{\perp}+(\Omega_0^{-1}B\Omega_0)\,\mathrm{d}t)DA^{\perp}\end{aligned}$$

(here the last step follows because $\mathcal{G}_{\Sigma}$ leaves the informal measure $DA^{\perp}$ on $\mathcal{A}^{\perp}$ invariant). This means that the function $\tilde{\chi}(B): C^{\infty}(\Sigma,\mathfrak{g})\ni B\mapsto\int\chi(A^{\perp}+B\,\mathrm{d}t)DA^{\perp}\in\mathbb{C}$ is

$\mathcal{G}_\Sigma$-invariant. Moreover, from Eq. (4.4) above we obtain for $B \in C^\infty(\Sigma, \mathfrak{g})$, $\Omega_0 \in \mathcal{G}_\Sigma$

$$\begin{aligned}\tilde{\Delta}[B] &= |\det(\partial/\partial t + \mathrm{ad}(B))| \stackrel{(*)}{=} |\det(\mathrm{Ad}(\Omega_0) \circ (\partial/\partial t + \mathrm{ad}(B)) \circ \mathrm{Ad}(\Omega_0^{-1}))| \\ &\stackrel{(+)}{=} |\det(\partial/\partial t + \mathrm{ad}(\mathrm{Ad}(\Omega_0) \cdot B))| = |\det(\partial/\partial t + \mathrm{ad}(B \cdot \Omega_0))| = \tilde{\Delta}[B \cdot \Omega_0]\end{aligned}$$

Here step $(*)$ follows because $\det(\mathrm{Ad}(g)) = 1$, $g \in G$ (as G is compact) and step $(+)$ follows because $\partial/\partial t$ commutes with $\mathrm{Ad}(\Omega_0)$ and because

$$\forall\, g \in G : B_0 \in \mathfrak{g} : \quad \mathrm{Ad}(g^{-1}) \circ \mathrm{ad}(B_0) \circ \mathrm{Ad}(g) = \mathrm{ad}(\mathrm{Ad}(g^{-1}) \cdot B_0) \tag{4.9}$$

Thus, the function $\tilde{\Delta}[\cdot]$ on $C^\infty(\Sigma, \mathfrak{g})$ is $\mathcal{G}_\Sigma$-invariant, too. By taking this into account and by computing the functional determinant of the covering $\tilde{\theta} : G/T \times \mathfrak{t}'_{\mathrm{reg}} \to \mathfrak{g}'_{\mathrm{reg}}$ given by $\tilde{\theta}((gT, B)) = g \cdot B \cdot g^{-1}$, $g \in G$, $B \in \mathfrak{t}'_{\mathrm{reg}}$, one obtains from Eq. (4.6)

$$\int_{\mathcal{A}} \chi(A) DA \sim \int_{C^\infty(\Sigma, \mathfrak{t}'_{\mathrm{reg}})} \left[\int_{\mathcal{A}^\perp} \chi(A^\perp + B\, \mathrm{d}t) DA^\perp \right] \tilde{\Delta}(B) DB \tag{4.10a}$$

where now

$$\begin{aligned} DB &= \det\left(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}(B))^n}{(n+1)!}\right) \cdot \det\left(-\mathrm{ad}(B)_{|\mathfrak{g}_0}\right) \mathrm{d}B \\ &\stackrel{(*)}{=} \det(\mathrm{id}_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0}))\, \mathrm{d}B \end{aligned} \tag{4.10b}$$

with $\mathrm{d}B$ denoting the "Lebesgue measure" on $C^\infty(\Sigma, \mathfrak{t})$. Here step $(*)$ holds because $\det(\sum_{n=0}^\infty (\mathrm{ad}(B))^n/(n+1)!) = \det(\sum_{n=0}^\infty (\mathrm{ad}(B)_{|\mathfrak{g}_0})^n/(n+1)!)$.

In fact, a more careful analysis shows that in order to derive (4.10a)–(4.10b) one has to make use of condition (iii) in Proposition 3.3. In particular, if Σ was assumed to be oriented then according to Proposition 3.4 this means that we have to demand additionally that Σ is *non-compact*. In order to demonstrate this let us concentrate for simplicity on the special case where G is semisimple. Then every connected component S^* of $\mathfrak{g}'_{\mathrm{reg}}$ is of the form $\mathrm{Ad}(G) \cdot P$, where P is a fixed alcove of $\mathfrak{t}'_{\mathrm{reg}}$. The restriction of $\tilde{\theta} : G/T \times \mathfrak{t}'_{\mathrm{reg}} \to \mathfrak{g}'_{\mathrm{reg}}$ onto the set $G/T \times P$ is a bijection onto the set S^*. This induces a bijection $\psi : C^\infty(\Sigma, G/T) \times C^\infty(\Sigma, P) \to C^\infty(\Sigma, S^*)$. Let $j : C^\infty(\Sigma, S^*) \to C^\infty(\Sigma, P)$ be given by $j := pr_2 \circ \psi^{-1}$ where $pr_2 : C^\infty(\Sigma, G/T) \times C^\infty(\Sigma, P) \to C^\infty(\Sigma, P)$ is the canonical projection.

Lemma 2. *If* $\mathrm{Image}(\Phi) \supset \mathcal{A}_{\mathrm{reg}}$ then $f = f \circ j$ *for every* $\mathcal{G}_\Sigma$*-invariant* $f \in C^\infty(\Sigma, S^*)$.

Proof. It is not difficult to see that $j(B \cdot \Omega_0) = j(B)$ for $B \in C^\infty(\Sigma, S^*)$, $\Omega_0 \in \mathcal{G}_\Sigma$ so it is enough to show that $f(B) = f(j(B))$ holds for all B in a complete system of representatives of $C^\infty(\Sigma, S^*)/\mathcal{G}_\Sigma$. The assumption on (Σ, G) that $\mathrm{Image}(\Phi) \supset \mathcal{A}_{\mathrm{reg}}$ holds and therefore also statement (iii) in Proposition 3.3 is fulfilled implies that $C^\infty(\Sigma, P)$ is a complete system of representatives of $C^\infty(\Sigma, S^*)/\mathcal{G}_\Sigma$. The assertion of Lemma 2 now follows from the fact that j restricted to $C^\infty(\Sigma, P)$ is just the identity. □

So if Image(Φ) $\supset \mathcal{A}_{\text{reg}}$ holds then using Lemma 2 and the $\mathcal{G}_\Sigma$-invariance of $\tilde{\Delta}$ and $\tilde{\chi}$ we obtain from Eq. (4.8)

$$\begin{aligned}\int_{\mathcal{A}} \chi(A) DA &\sim \int_{C^\infty(\Sigma,S^*)} \left[\int_{\mathcal{A}^\perp} \chi(A^\perp + B\,\mathrm{d}t) DA^\perp\right] \tilde{\Delta}(B) D_{S^*} B \\ &= \int_{C^\infty(\Sigma,S^*)} \tilde{\chi}(B)\tilde{\Delta}(B) D_{S^*} B = \int_{C^\infty(\Sigma,S^*)} (\tilde{\chi}\circ j)(B)(\tilde{\Delta}\circ j)(B) D_{S^*} B \\ &= \int_{C^\infty(\Sigma,P)} \tilde{\chi}(B)\, \tilde{\Delta}(B) D_P B \\ &= \int_{C^\infty(\Sigma,P)} \left[\int_{\mathcal{A}^\perp} \chi(A^\perp + B\,\mathrm{d}t) DA^\perp\right] \tilde{\Delta}(B) D_P B \end{aligned} \tag{4.11}$$

where $D_P B$ denotes the image measure $j_*(D_{S^*}B)$ of $D_{S^*}B$ under j. One has

$$\begin{aligned} D_P B &= j_*\left(\det\left(\sum_{n=0}^\infty \frac{(\mathrm{ad}(B))^n}{(n+1)!}\right) 1_{C^\infty(\Sigma,S^*)}(B)\mathrm{d}B\right) \\ &\overset{(*)}{=} \det\left(\sum_{n=0}^\infty \frac{\mathrm{ad}(B)^n}{(n+1)!}\right) 1_{C^\infty(\Sigma,P)}(B)\det(-\mathrm{ad}(B)_{|\mathfrak{g}_0})\mathrm{d}B \end{aligned} \tag{4.12}$$

where the last $\mathrm{d}B$ denotes the heuristic Lebesgue measure on $C^\infty(\Sigma,\mathfrak{t})$. Here step $(*)$ follows from $j_*(\mathrm{d}B) = \det(-\mathrm{ad}(B)_{|\mathfrak{g}_0})\,\mathrm{d}B$ and $\det(\sum_{n=0}^\infty (\mathrm{ad}(B))^n/(n+1)!) = \det(\sum_{n=0}^\infty \mathrm{ad}(j(B))^n/(n+1)!)$ (the latter equation follows easily from Eq. (4.9) above). Using an infinite averaging procedure over all alcoves P one finally arrives at (4.10a) and (4.10b).

5. Chern-Simons models

5.1. Basic definitions

In this subsection, M will denote an arbitrary compact, connected, and oriented 3-manifold. Let us fix $k \in \mathbb{Z}\backslash\{0\}$ and set $\lambda := 1/k$. The function

$$S_{CS} : \mathcal{A} \ni A \mapsto \frac{k}{4\pi}\int_M \mathrm{Tr}_{\mathrm{Mat}(N,\mathbb{C})}\left(A\wedge\mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right) \in \mathbb{C} \tag{5.1}$$

will be called "the action function of the pure Chern-Simons model on M with structure group G and charge k". Here we have embedded $\mathcal{A}$ into the space $\mathcal{A}_{\mathrm{Mat}(N,\mathbb{C})}$ of all smooth Mat(N, $\mathbb{C}$)-valued 1-forms on M (recall that $\mathfrak{g} \subset u(N) \subset \mathrm{Mat}(N,\mathbb{C})$). In particular, $\wedge$ denotes the wedge product of $\mathcal{A}_{\mathrm{Mat}(N,\mathbb{C})}$.

Clearly, S_{CS} is invariant under orientation-preserving diffeomorphisms. It has been suggested by Witten [32] (see also, e.g., [6]) that if one can make sense of the heuristic measure

$$\mu_{CS}(\mathrm{d}A) := \frac{1}{Z}\exp(iS_{CS}(A))DA \tag{5.2}$$

where "DA" is the heuristic "Lebesgue measure" on $\mathcal{A}$ and "Z" the normalization constant "$\int \exp(iS_{CS}(A))DA$" one can obtain non-trivial link invariants by integrating certain functions on $\mathcal{A}$ against μ_{CS}. More precisely, for a given link L in M, i.e., a tuple $(l_1, \ldots, l_n)$, $n \in \mathbb{N}$, of loops in M whose arcs are pairwise disjoint (a loop being a smooth embedding of S^1 into M) let us consider the function $\mathrm{WLF}(L) : \mathcal{A} \ni A \mapsto \prod_{i=1}^{n} \mathrm{Tr}(\mathrm{Hol}(A; l_i)) \in \mathbb{C}$ where $\mathrm{Hol}(A; l)$ denotes the holonomy of A around l. Due to the diffeomorphism invariance of S_{CS} and hence also ofμ_{CS}, the heuristic integral $\mathrm{WLO}(L) := \int \mathrm{WLF}(L)\, \mathrm{d}\mu_{CS}$, the so-called "Wilson loop observable associated to the link L", should depend only on the isotopy class of L. So the mapping which maps every (sufficiently regular) link L to $\mathrm{WLO}(L)$ should be a link invariant. According to the standard literature in the special case $M = S^3$ and $G = SU(N)$ (resp. $SO(N)$) this link invariant should be related to the Homfly (resp. the Kauffman) polynomial, cf. [24].

Remark 5.1. From $k \in \mathbb{Z}$ it follows that $\exp(iS_{CS})$ is gauge invariant even though S_{CS} itself is not (cf., e.g., [32,16]).

5.2. Chern-Simons models on $M = \Sigma \times S^1$

Now we restrict our attention to Chern-Simons models on $M = \Sigma \times S^1$, where Σ is a compact oriented surface.

Proposition 5.1. *Let $A \in \mathcal{A}$ and let $A^{\perp} \in \mathcal{A}^{\perp}$ and $A^{||} \in \mathcal{A}^{||}$ be given by $A = A^{\perp} + A^{||}$. Then we have*

$$S_{CS}(A) = \frac{k}{4\pi}\left[\int \mathrm{Tr}(A^{\perp} \wedge \mathrm{d}A^{\perp}) + 2\int \mathrm{Tr}(A^{\perp} \wedge A^{||} \wedge A^{\perp}) + 2\int \mathrm{Tr}(A^{\perp} \wedge \mathrm{d}A^{||})\right] \tag{5.3}$$

Proof. We have $A \wedge A \wedge A = (A^{\perp} + A^{||}) \wedge (A^{\perp} + A^{||}) \wedge (A^{\perp} + A^{||}) = A^{\perp} \wedge A^{||} \wedge A^{\perp} + A^{||} \wedge A^{\perp} \wedge A^{\perp} + A^{\perp} \wedge A^{\perp} + A^{||}$ (the other 5 summands clearly vanish). Thus, we obtain $(2/3)\mathrm{Tr}(A \wedge A \wedge A) = (2/3) \cdot 3 \cdot \mathrm{Tr}(A^{\perp} \wedge A^{||} \wedge A^{\perp})$. On the other hand $A \wedge \mathrm{d}A = (A^{\perp} + A^{||}) \wedge (\mathrm{d}A^{\perp} + \mathrm{d}A^{||}) = (A^{\perp} \wedge \mathrm{d}A^{\perp}) + (A^{||} \wedge \mathrm{d}A^{\perp}) + (A^{\perp} \wedge \mathrm{d}A^{||}) + 0$. But $\mathrm{d}(A^{\perp} \wedge A^{||}) = \mathrm{d}A^{\perp} \wedge A^{||} - A^{\perp} \wedge \mathrm{d}A^{||}$ and thus from Stokes Theorem we have $0 = \int \mathrm{d}(\mathrm{Tr}(A^{\perp} \wedge A^{||})) = \int \mathrm{Tr}(\mathrm{d}A^{\perp} \wedge A^{||}) - \int \mathrm{Tr}(A^{\perp} \wedge \mathrm{d}A^{||})$. If one takes into account that $\mathrm{Tr}(A^{||} \wedge \mathrm{d}A^{\perp}) = \mathrm{Tr}(\mathrm{d}A^{\perp} \wedge A^{||})$ one finally obtains (5.3). □

Definition 5.1. For every real vector space V and $k \in \{1, 2\}$ we will denote the space of V-valued k-forms on Σ by $\Omega^k(\Sigma, V)$. We will call a function $\alpha : S^1 \to \Omega^k(\Sigma, V)$ *smooth* if for every k-tuple $(X_i)_{i\leq k}$ of C^{∞}-vector fields on Σ the function $\Sigma \times S^1 \ni (\sigma, t) \mapsto (\alpha(t)((X_i)_{i\leq k}))_{\sigma} \in V$ is C^{∞}. The space of all smooth functions $S^1 \to \Omega^k(\Sigma, V)$ will be denoted by $C^{\infty}(S^1, \Omega^k(\Sigma, V))$. The mapping $C^{\infty}(S^1, \Omega^k(\Sigma, V)) \ni \alpha \mapsto \partial/\partial t\alpha \in C^{\infty}(S^1, \Omega^k(\Sigma, V))$ where $\partial/\partial t\alpha$ is given by $\partial/\partial t[\alpha(t)((X_i)_{i\leq k})_{\sigma}] = (\partial/\partial t\alpha)(t)((X_i)_{i\leq k})_{\sigma}$ for every k-tuple $(X_i)_{i\leq k}$ of C^{∞}-vector fields on Σ and every $\sigma \in \Sigma$ will be denoted by $\partial/\partial t$. Finally, we will set $\Omega^k(\Sigma) := \Omega^k(\Sigma, \mathbb{C})$, $\mathcal{A}_{\Sigma,V} := \Omega^1(\Sigma, V)$, and $\mathcal{A}_{\Sigma} := \mathcal{A}_{\Sigma,\mathfrak{g}} = \Omega^1(\Sigma, \mathfrak{g})$.

During the rest of this paper we will identify $\mathcal{A}^{\perp}$ with $C^{\infty}(S^1, \mathcal{A}_{\Sigma})$ in the obvious way. In particular, if $A^{\perp} \in \mathcal{A}^{\perp}$ and $t \in S^1$ then $A^{\perp}(t)$ will denote an element of $\mathcal{A}_{\Sigma}$.

Proposition 5.2. *Let $A \in \mathcal{A}^{qax}$ and let $A^{\perp} \in \mathcal{A}^{\perp} \cong C^{\infty}(S^1, \mathcal{A}_{\Sigma})$ and $B \in C^{\infty}(\Sigma, \mathfrak{g})$ be given by $A = A^{\perp} + B\,\mathrm{d}t$. Then we have*

$$\begin{aligned} S_{CS}(A) &= S_{CS}(A^{\perp} + B\,\mathrm{d}t) \\ &= -\frac{k}{4\pi}\int_{S^1} \mathrm{d}t \left[\left\langle A^{\perp}(t), (\partial/\partial t + \mathrm{ad}(B)) \cdot A^{\perp}(t)\right\rangle_{\Sigma} - 2\langle A^{\perp}(t), \mathrm{d}B\rangle_{\Sigma}\right] \end{aligned} \tag{5.4}$$

where $\langle\cdot,\cdot\rangle_{\Sigma}$ denotes the bilinear form on $\mathcal{A}_{\Sigma}$ given by $\langle A, A'\rangle_{\Sigma} := \int_{\Sigma} \mathrm{Tr}(A \wedge A')$ for $A, A' \in \mathcal{A}_{\Sigma} \subset \mathcal{A}_{\Sigma,\mathrm{Mat}(N,\mathbb{C})}$.

Proof. For every $\alpha \in C^{\infty}(S^1, \Omega^2(\Sigma))$ let $i(\alpha)$ denote the complex 2-form on $\Sigma \times S^1$ induced by α. Then we have $\mathrm{Tr}(A^{\perp} \wedge \mathrm{d}A^{||}) = \mathrm{Tr}(A^{\perp} \wedge d(B\,\mathrm{d}t)) = \mathrm{Tr}(A^{\perp} \wedge \mathrm{d}B) \wedge \mathrm{d}t = i(\alpha_1) \wedge \mathrm{d}t$ with $\alpha_1 \in C^{\infty}(S^1, \Omega^2(\Sigma))$ given by $\alpha_1(t) = \mathrm{Tr}[A^{\perp}(t) \wedge \mathrm{d}B]$, $t \in S^1$. On the other hand $2\mathrm{Tr}(A^{\perp} \wedge A^{||} \wedge A^{\perp}) = 2\mathrm{Tr}(A^{\perp} \wedge B\,\mathrm{d}t \wedge A^{\perp}) = -\mathrm{Tr}(A^{\perp} \wedge (\mathrm{ad}(B) \cdot A^{\perp})) \wedge \mathrm{d}t = i(\alpha_2) \wedge \mathrm{d}t$ with $\alpha_2 \in C^{\infty}(S^1, \Omega^2(\Sigma))$ given by $\alpha_2(t) = -\mathrm{Tr}(A^{\perp}(t) \wedge (\mathrm{ad}(B) \cdot A^{\perp}(t)))$, $t \in S^1$. Finally, using local coordinates it is not difficult to show that $\mathrm{Tr}(A^{\perp} \wedge \mathrm{d}A^{\perp}) = i(\alpha_3) \wedge \mathrm{d}t$ with $\alpha_3 \in C^{\infty}(S^1, \Omega^2(\Sigma))$ given by $\alpha_3(t) = -\mathrm{Tr}(A^{\perp}(t) \wedge (\partial/\partial t)A^{\perp}(t)))$, $t \in S^1$. The assertion of the proposition now follows immediately from the following Lemma, which is easy to prove. □

Lemma 3. *For every $\alpha \in C^{\infty}(S^1, \Omega^2(\Sigma))$ the mapping $S^1 \ni t \mapsto \int_{\Sigma} \alpha(t) \in \mathbb{C}$* is *$C^{\infty}$ and we have $\int_{\Sigma \times S^1} i(\alpha) \wedge \mathrm{d}t = \int_{S^1} [\int_{\Sigma} \alpha(t)]\,\mathrm{d}t$*

6. Quasi-axial gauge fixing and torus gauge fixing for Chern-Simons models on $\Sigma \times S^1$

6.1. *Application of quasi-axial gauge fixing*

Let Σ be as in Section 5.2. If we assume additionally that Σ or G is simply-connected then we can make use of (4.6) and the gauge invariance of $\mathrm{WLF}(L)$ and of $\exp(iS_{CS})$ (cf. Remark 5.1) and we then obtain informally, with $\sim$ denoting equality up to a multiplicative constant, independent of L,

$$\begin{aligned} WLO(L) &= \int WLF(L)(A)\frac{1}{Z}\exp(iS_{CS}(A))DA \sim \int_{C^{\infty}(\Sigma,\mathfrak{g}'_{\mathrm{reg}})} \\ &\quad \times \int_{\mathcal{A}^{\perp}} WLF(L)(A^{\perp} + B\,\mathrm{d}t)\exp(iS_{CS}(A^{\perp} + B\,\mathrm{d}t))DA^{\perp}\tilde{\Delta}[B]DB \\ &= \int_{C^{\infty}(\Sigma,\mathfrak{g}'_{\mathrm{reg}})} \left[\int_{\mathcal{A}^{\perp}} WLF(L)(A^{\perp} + B\,\mathrm{d}t)\,\mathrm{d}\mu_B^{\perp}(A^{\perp})\right]\tilde{\Delta}[B]DB \end{aligned} \tag{6.1}$$

where $DA^{\perp}$ and DB are as in the Subsection 4.2 and $\mu_B^{\perp}$, for $B \in C^{\infty}(\Sigma, \mathfrak{g}'_{\mathrm{reg}})$, is given informally by $\mathrm{d}\mu_B^{\perp}(A^{\perp}) := \exp(iS_{CS}(A^{\perp} + B\,\mathrm{d}t))DA^{\perp}$.

According to Proposition 5.2, for fixed $B \in C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}})$ the function $S_{CS}(A^\perp + B\,\mathrm{d}t)$ on $\mathcal{A}^\perp$ is quadratic so the informal measure $\mu_B^\perp$ on $\mathcal{A}^\perp$ is of "Gauss-type". If one tries to compute the informal "mean" and "covariance operator" of $\mu_B^\perp$ then, at least[3] for Abelian G, one is naturally lead to the decomposition $\mathcal{A}^\perp = \hat{\mathcal{A}}^\perp \oplus \mathcal{A}_c^\perp$, where $\hat{\mathcal{A}}^\perp := \{A^\perp \in \mathcal{A}^\perp \mid A^\perp(t_0) = 0\}$, $\mathcal{A}_c^\perp := \{A^\perp \in \mathcal{A}^\perp \mid A^\perp(t) = A^\perp(t_0) \quad \forall t \in S^1\} \cong \mathcal{A}_\Sigma$, cf. Remark 6.1, Remark 6.2, and Remark 8.2 below. It is not difficult to see that $S_{CS}(A^\perp + B\,\mathrm{d}t) = S_{CS}(\hat{A}^\perp + B\,\mathrm{d}t) - (k/4\pi)[\int_{S^1} \mathrm{d}t \langle \hat{A}^\perp(t), \mathrm{ad}(B) \cdot A_c^\perp \rangle_\Sigma + \langle A_c^\perp, \mathrm{ad}(B) \cdot A_c^\perp \rangle_\Sigma - 2\langle A_c^\perp, \mathrm{d}B \rangle_\Sigma]$ so if one introduces $\hat{Z}(B) = \int \exp(iS_{CS}(\hat{A}^\perp + B\,\mathrm{d}t))D\hat{A}^\perp$ and

$$\mathrm{d}\hat{\mu}_B^\perp(\hat{A}^\perp) := \frac{1}{\hat{Z}(B)} \exp(iS_{CS}(\hat{A}^\perp + B\,\mathrm{d}t))D\hat{A}^\perp \tag{6.2}$$

where $D\hat{A}^\perp$ denotes the informal Lebesgue measure on $\hat{\mathcal{A}}^\perp$ one obtains from (6.1)

$$\begin{aligned}\mathrm{WLO}(L) = \frac{1}{Z'} \int_{C^\infty(\Sigma, \mathfrak{g}'_{\mathrm{reg}})} &\Bigg[\int_{\mathcal{A}_c^\perp} \Bigg[\int_{\hat{\mathcal{A}}^\perp} WLF(L)(\hat{A}^\perp + A_c^\perp + B\,\mathrm{d}t) \\ &\times \exp\left(-i\frac{k}{4\pi} \int_{S^1} \mathrm{d}t \langle \hat{A}^\perp(t), \mathrm{ad}(B) A_c^\perp \rangle_\Sigma\right) \mathrm{d}\hat{\mu}_B^\perp(\hat{A}^\perp)\Bigg] \\ &\times \exp\left(i\frac{k}{2\pi} \langle A_c^\perp, \mathrm{d}B \rangle_\Sigma\right) \times \exp\left(-i\frac{k}{4\pi} \langle A_c^\perp, \mathrm{ad}(B) A_c^\perp \rangle_\Sigma\right) DA_c^\perp\Bigg] \\ &\tilde{\Delta}[B]\hat{Z}(B)DB\end{aligned} \tag{6.3}$$

where the normalization constant Z' is given by the integral expression obtained from the right-hand side of (6.3) by replacing the function $\mathrm{WLF}(L)(\hat{A}^\perp + A_c^\perp + B\,\mathrm{d}t)$ by the constant function taking only the value 1.

Let us now consider for a while the special case where G is Abelian. We have assumed above that Σ or G is simply-connected. G can not be simply-connected if it is Abelian so we are forced to restrict ourselves to the case $\Sigma \cong S^2$. Furthermore, for Abelian G we have $G = G_{\mathrm{reg}}$, $\mathfrak{g} = \mathfrak{g}'_{\mathrm{reg}}$, $\mathrm{ad}(B)A_c^\perp = 0$ and $\hat{Z}(B)\tilde{\Delta}[B]$ does not depend on B which implies $Z' = \hat{Z}(0)\tilde{\Delta}[0] \int \exp(i(k/2\pi)\langle A_c^\perp, \mathrm{d}B \rangle_\Sigma) DA_c^\perp DB$. Thus, (6.3) simplifies and we obtain

$$\begin{aligned}\mathrm{WLO}(L) = \frac{1}{Z''} \int_{C^\infty(\Sigma, \mathfrak{g})} &\Bigg[\int_{\mathcal{A}_c^\perp} \Bigg[\int_{\hat{\mathcal{A}}^\perp} WLF(L)(\hat{A}^\perp + A_c^\perp + B\,\mathrm{d}t)\mathrm{d}\hat{\mu}_B^\perp(\hat{A}^\perp)\Bigg] \\ &\times \exp\left(i\frac{k}{2\pi} \langle A_c^\perp, \mathrm{d}B \rangle_\Sigma\right) DA_c^\perp\Bigg] DB\end{aligned} \tag{6.4}$$

where $Z'' = \int \int \exp(ik/2\pi \langle A_c^\perp, \mathrm{d}B \rangle_\Sigma) DA_c^\perp DB$.

In Sections 7–10, we will show how one can make sense of the right-hand side of (6.4).

[3] If G is non-Abelian then it is possibly better to use the more complicated decomposition which is mentioned in the last paragraph of Section 6.1

Remark 6.1. The reader may wonder why we chose to use the measure $\hat{\mu}^{\perp}_B$ on $\hat{\mathcal{A}}^{\perp}$ instead of the measure $\mu^{\perp}_B$ on $\mathcal{A}^{\perp}$ even though Eqs. (6.3) and (6.4) look more complicated than Eq. (6.1). From a computational point of view the answer is that it is only because $\hat{\mu}^{\perp}_B$ "lives" on $\hat{\mathcal{A}}^{\perp}$ that we can identify its "mean" $m(B)$ and its "covariance operator" $C(B)$. E.g., in the special case where G is Abelian or where G is non-Abelian and $B \in C^{\infty}(\Sigma, \mathfrak{t}'_{\text{reg}})$, $m(B)$ is given by Eq. (8.7) below and $C(B)$ is given by Eq. (8.5) in Proposition 8.1 combined with Eq. (8.3). Note that Eqs. (8.7) and (8.3) are essential for the proof of Theorem 10.1: (8.7) is used in (10.4) and leads to the appearance of the expressions $\text{sgn}(l^j_{S^1}; u)$ in Eq. (10.1) in Theorem 10.1. Eq. (8.3) is used in (10.12) and later leads to the appearance of the linking number expressions in Eq. (10.1). Closely related to this computational advantage is the conceptual advantage that – based on the explicit expressions for $m(B)$ and $C(B)$ – it is possible to give a rigorous meaning to the integral functional $\int \cdots d\hat{\mu}^{\perp}_B$, at least for the special case $B \in C^{\infty}(\Sigma, \mathfrak{t}'_{\text{reg}})$, cf. Section 8.2 below.

Remark 6.2. One could try to avoid the decomposition $\mathcal{A}^{\perp} = \hat{\mathcal{A}}^{\perp} \oplus \mathcal{A}^{\perp}_c$ or to use the decomposition $\mathcal{A}^{\perp} = \{A^{\perp} \in \mathcal{A}^{\perp} \mid \int_{S^1} A^{\perp}(t)\, dt = 0\} \oplus \mathcal{A}^{\perp}_c$ instead, i.e., the decomposition of $\mathcal{A}^{\perp}$ into "zero-modes" and "non-zero modes". The second option would at first look have the advantage that also for non-Abelian G the term $\exp(-ik/4\pi \int_{S^1} dt \langle \hat{A}^{\perp}(t), \text{ad}(B) \cdot A^{\perp}_c \rangle_{\Sigma})$ in the inner integral in (6.3) vanishes. However, one would still be forced to consider the space $\hat{\mathcal{A}}^{\perp}$ later, cf. Remark 8.2 below. Moreover, one would then have to insert the expression $\exp(-i(k/2\pi)\langle \hat{A}^{\perp}(t_0), dB \rangle_{\Sigma})$ into the inner integral in (6.3) which is clearly more singular than $\exp(-i(k/2\pi) \int_{S^1} dt \langle \hat{A}^{\perp}(t), \text{ad}(B) \cdot A^{\perp}_c \rangle_{\Sigma})$.

Also for non-Abelian G it should be possible to find a rigorous realization of the integral functional $\int \cdots d\hat{\mu}^{\perp}_B$ by modifying the approach in Section 8 below. For example, the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_0$ which is used in Section 8 will have to depend on σ (for each σ, $\mathfrak{t}$ will have to be replaced by the maximal Abelian Lie subalgebra which contains $B(\sigma)$). By using this more complicated decomposition one obtains a version of Eq. (6.3) in which no terms like $\int_{S^1} dt \langle \hat{A}^{\perp}(t), \text{ad}(B) \cdot A^{\perp}_c \rangle_{\Sigma}$ and $\langle A^{\perp}_c, \text{ad}(B) \cdot A^{\perp}_c \rangle_{\Sigma}$ appear. Accordingly, this modified version of Eq. (6.3) will look very similar to Eq. (6.6) below.

6.2. Application of torus gauge fixing

Before one studies the integral functional $\int \cdots d\hat{\mu}^{\perp}_B$ in more detail also for non-Abelian G it is reasonable to ask first whether by using torus gauge fixing things can be simplified. As above let us assume that Σ is a compact oriented surface and Σ or G is simply-connected. Then, according to Proposition 3.4, the relation $\text{Image}(\Phi) \supset \mathcal{A}_{\text{reg}}$, which was used for the derivation of Eqs. (4.10a)–(4.10b) in Section 4, does not hold. Anyhow, let us study what happens if we assume that (4.10a)–(4.10b) still hold. Under this assumption one can easily derive a "torus gauge analogue" of Eq. (6.1) by replacing $C^{\infty}(\Sigma, \mathfrak{g}'_{\text{reg}})$ by $C^{\infty}(\Sigma, \mathfrak{t}'_{\text{reg}})$. In particular, the measure DB will then be the measure of Section 4.3. Let us now introduce a "new" decomposition $\mathcal{A}^{\perp} = \hat{\mathcal{A}}^{\perp} \oplus \mathcal{A}^{\perp}_c$ by setting

$$\hat{\mathcal{A}}^{\perp} := \{A^{\perp} \in \mathcal{A}^{\perp} \mid \pi_{\mathcal{A}_{\Sigma,\mathfrak{t}}}(A^{\perp}(t_0)) = 0\} \tag{6.5a}$$

$$\mathcal{A}^{\perp}_c := \{A^{\perp} \in \mathcal{A}^{\perp} \mid A^{\perp}(t) = A^{\perp}(t_0) \in \mathcal{A}_{\Sigma,\mathfrak{t}} \quad \forall t \in S^1\} \cong \mathcal{A}_{\Sigma,\mathfrak{t}} \tag{6.5b}$$

Here $\pi_{\mathcal{A}_{\Sigma,\mathfrak{t}}}$ is the projection operator onto the second term in the direct sum $\mathcal{A}_\Sigma \cong \mathcal{A}_{\Sigma,\mathfrak{g}_0} \oplus \mathcal{A}_{\Sigma,\mathfrak{t}}$ where $\mathfrak{g}_0$ is as in Subsec. 4.3. Note that if $A_c^\perp$ is an element of the "new" space $\mathcal{A}_c^\perp$ then $\mathrm{ad}(B) \cdot A_c^\perp = 0$. So we obtain the following "torus gauge analogue" of Eq. (6.3)

$$\mathrm{WLO}(L) = \frac{1}{Z} \int_{C^\infty(\Sigma,\mathfrak{t}'_{\mathrm{reg}})} \left[\int_{\mathcal{A}_c^\perp} \left[\int_{\hat{\mathcal{A}}^\perp} WLF(L)(\hat{A}^\perp + A_c^\perp + B\,\mathrm{d}t)\mathrm{d}\hat{\mu}_B^\perp(\hat{A}^\perp) \right] \right.$$
$$\left. \times \exp\left(i\frac{k}{2\pi} \langle A_c^\perp, \mathrm{d}B\rangle_\Sigma \right) DA_c^\perp \right] \tilde{\Delta}[B]\hat{Z}(B)DB \qquad (6.6)$$

where $\hat{Z}(B)$ and $\hat{\mu}_B^\perp$ are defined in a similar way as above (cf. (6.2)) and where Z is given by

$$Z = \int_{C^\infty(\Sigma,\mathfrak{t}'_{\mathrm{reg}})} \int_{\mathcal{A}_c^\perp} \exp\left(i\frac{k}{2\pi} \langle A_c^\perp, \mathrm{d}B\rangle_\Sigma \right) \tilde{\Delta}[B]\hat{Z}(B)DA_c^\perp DB \qquad (6.7)$$

Below we will see that for $B \in C^\infty(\Sigma, \mathfrak{t}'_{\mathrm{reg}})$ one has informally $\hat{Z}(B) \sim |\det(\partial/\partial t + \mathrm{ad}(B))|^{-1/2}$ (with $\sim$ denoting equality up to a multiplicative constant independent of B), so

$$\hat{Z}(B)\tilde{\Delta}[B] \sim \frac{|\det(\partial/\partial t + \mathrm{ad}(B))|}{|\det(\partial/\partial t + \mathrm{ad}(B))|^{1/2}} \qquad (6.8)$$

where the operator $\partial/\partial t + \mathrm{ad}(B)$ in the numerator is defined on $C^\infty_{\mathfrak{g}}(\Sigma \times S^1)$ and the operator $\partial/\partial t + \mathrm{ad}(B)$ in the denominator is defined on $C^\infty(S^1, \mathcal{A}_\Sigma)$. If we compare Eq. (6.8) with Eqs. (2.11) and (2.17) in [10] we see that the right-hand side of Eq. (6.7) coincides with Eq. (2.18) in [10] (cf. also Eq. (6.33) in [10] which shows that the first fraction in Eq. (2.11) in [10] is just a constant). By evaluating Eq. (2.18) in [10] explicitly Blau and Thompson finally obtained the Verlinde formula (7.13) in [10]. However, in order to derive their Eq. (7.13) they had to insert certain "correction terms", given by Eq. (7.7) in [10], into Eq. (7.5) in [10]. Later, in [12], they showed that these correction terms can be explained naturally as those terms that appear if one modifies Eq. (7.5) in [10] in such a way that it takes into account all the possible torus bundles on Σ and not only the trivial ones, cf. Eq. (6.8) in [12] (cf. also our Remark 3.3 above). It is reasonable to expect that also for Eq. (6.6) above similar correction terms have to be considered. We expect that these correction terms will only affect the outer integrations in (6.6). In the present paper, we will mainly be concerned with the inner integration in Eq. (6.6) (below we will study the outer integrations only for Abelian G for which there are no correction terms). For this reason we will postpone the search for the correct form of these correction terms to a forthcoming paper, see [18].

Remark 6.3. So far we have only considered the case where Σ is compact. However, it is straightforward to extend the framework given in Section 5 so that also non-compact oriented surfaces Σ can be treated, e.g., by replacing the space $\mathcal{A}$ by the space of 1-forms on M with compact support and so on. According to Proposition 3.4, for non-compact oriented surfaces Σ the three equivalent conditions in Proposition 3.3 hold. Thus, non-compact surfaces Σ have the advantage that one can actually "derive" (a non-compact version of) Eq. (6.6) at an informal level. On the other hand non-compact surfaces also have several disadvantages. For example, it is not clear whether the condition $k \in \mathbb{Z}$ in Remark 5.1 has to be replaced by a different condition, cf. [21]. By studying the non-compact version of (6.6) and the implications for the values of the WLOs we hope to be able to shed some new light on this point.

7. Some Notions and Results from White Noise Analysis

Let $\mathcal{H}$ be a real separable Hilbert space with norm $\|\cdot\|$. Let $\mathcal{K}$ be a self-adjoint invertible Hilbert–Schmidt operator on $\mathcal{H}$ whose Hilbert–Schmidt norm is strictly less than 1. We define $\mathcal{N}_p := \text{Image}(\mathcal{K}^p)$, $p \in \mathbb{N}_0$ and $\mathcal{N} := \bigcap_{p\in\mathbb{N}_0} \mathcal{N}_p$ and introduce the norms $\|\cdot\|_p := \|\mathcal{K}^{-p}(\cdot)\|$, $p \in \mathbb{N}_0$, on $\mathcal{N}$. Then we equip the space $\mathcal{N}$ with the topology which is generated by the family $(\|\cdot\|_p)_{p\in\mathbb{N}_0}$. We denote the topological dual of $\mathcal{N}$ by $\mathcal{N}^*$.

According to the Minlos Theorem there is a unique Borel probability measure μ on $\mathcal{N}^*$ with the property that for all $x \in \mathcal{N}$ the function $\mathcal{N}^* \ni T \mapsto (T, x) \in \mathbb{R}$ is a real Gaussian random variable with mean 0 and variance $\|x\|^2$. Here and in the sequel $(\cdot,\cdot)$ is the canonical pairing between $\mathcal{N}^*$ and $\mathcal{N}$. For every $p \in \mathbb{N}_0$, $\mathcal{K}^{-p}$ induces a (densely defined) operator $\Gamma(\mathcal{K}^{-p})$ on $L^2(\mathcal{N}^*, \mu)$ in a natural way, the so-called "second quantization" of $\mathcal{K}^{-p}$ (see Section 3 C in [22]).

By $\mathcal{P}(\mathcal{N})$ (resp. $\mathcal{E}(\mathcal{N})$) we denote the subalgebra of $C_{\mathbb{C}}(\mathcal{N}^*)$ generated by the subset $\{(\cdot, x) \mid x \in \mathcal{N}\}$ (resp. the set $\{\exp(i(\cdot, x)) \mid x \in \mathcal{N}\}$) of $C_{\mathbb{C}}(\mathcal{N}^*)$. We identify $\mathcal{P}(N)$ and $\mathcal{E}(\mathcal{N})$ with the obvious subspaces of $L^2(\mathcal{N}^*, \mu)$. It can be shown (see section 3 C in [22]) that $\mathcal{P}(\mathcal{N})$ is in the domain of all the operators $\Gamma(\mathcal{K}^{-p})$, so we can define scalar products $\langle\langle\cdot,\cdot\rangle\rangle_p$ on $\mathcal{P}(\mathcal{N})$ by $\langle\langle\phi, \phi'\rangle\rangle_p := \langle\langle\Gamma(\mathcal{K}^{-p})\phi, \Gamma(\mathcal{K}^{-p})\phi'\rangle\rangle$ for every $\phi, \phi' \in \mathcal{P}(\mathcal{N})$, where $\langle\langle\cdot,\cdot\rangle\rangle$ is the scalar product on $L^2(\mathcal{N}^*, \mu)$. We denote the norm associated to $\langle\langle\cdot,\cdot\rangle\rangle_p$ by $\|\cdot\|_p$ and the completion of $\mathcal{P}(\mathcal{N})$ w.r.t. $\|\cdot\|_p$ by $(\mathcal{N})_p$. The extended norm on $(\mathcal{N})_p$ will again be denoted by $\|\cdot\|_p$. Moreover, we identify the space $(\mathcal{N})_0$ with $L^2_{\mathbb{C}}(\mathcal{N}^*, \mu)$ in the obvious way and the spaces $(\mathcal{N})_p$, $p \in \mathbb{N}$, with the obvious subspaces of $(\mathcal{N})_0$. Then we set $(\mathcal{N}) := \bigcap_p (\mathcal{N})_p$ and equip $(\mathcal{N})$ with the topology which is generated by the family $(\|\cdot\|_p)_{p\in\mathbb{N}_0}$. The topological dual of $(\mathcal{N})$ will be denoted by $(\mathcal{N})^*$. It is not difficult to see that $\mathcal{E}(\mathcal{N}) \subset (\mathcal{N})$.

Theorem 7.1. *For every continuous quadratic form* Q on $\mathcal{N}$ *and every continuous linear form* a on $\mathcal{N}$ *there is a unique element* $\Phi_{a,Q}$ of $(\mathcal{N})^*$ *such that* $\Phi_{a,Q}(\exp(i(\cdot, f))) = \exp(ia(f))\exp(-(1/2)Q(f))$ *holds for all* $f \in \mathcal{N}$.

Proof. It can be shown that the map $\mathcal{N} \ni f \mapsto \exp(ia(f))\exp(-(1/2)Q(f)) \in \mathbb{C}$ is a "U-functional" in the terminology of [22]. From Theorem 4.38 in [22] the assertion follows. □

Remark 7.1. We will call $\Phi_{a,Q}$ "the Gaussian element of $(\mathcal{N})^*$ with mean a and covariance Q" or simply "the Gaussian element of $(\mathcal{N})^*$ corresponding to (a, Q)". By definition, $\Phi_{a,Q}$ is the unique element Φ of $(\mathcal{N})^*$ with the property that the mapping $\mathcal{N} \ni f \mapsto \Phi(\exp(i(\cdot, f))) \in \mathbb{C}$, called the "$\mathcal{T}$-transform of Φ" equals $\exp(ia(\cdot))\exp(-(1/2)Q(\cdot))$. Note that the $\mathcal{T}$-transform on $(\mathcal{N})^*$ can be considered as a generalization of the Fourier transformation on the space of bounded Borel measures on $\mathcal{N}^*$.

8. Rigorous implementation of the integral functional $\int \cdots d\hat{\mu}_B^{\perp}$ in (6.4) and (6.6)

In the present section, we will make rigorous sense of the integral functional $\int \cdots d\hat{\mu}_B^{\perp}$, $B \in C^{\infty}(\Sigma, \mathfrak{t}'_{\mathrm{reg}})$, in Eq. (6.6) as a generalized distribution on a suitable extension $\overline{\hat{\mathcal{A}}^{\perp}}$ of the space $\hat{\mathcal{A}}^{\perp}$ given by (6.5a). As for Abelian G the two Eqs (6.6) and (6.4) coincide we will then also have made rigorous sense of the integral functional $\int \cdots d\hat{\mu}_B^{\perp}$ in (6.4).

8.1. Some Preparations

Recall that $\mathfrak{g}_0$ denotes the $(\cdot,\cdot)_{\mathfrak{g}}$-orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$. As $(\cdot,\cdot)_{\mathfrak{g}}$ is Ad-invariant $\mathfrak{g}_0$ is $\mathrm{ad}_{|\mathfrak{t}}$-invariant. So if we fix $B_0 \in \mathfrak{t}'_{\mathrm{reg}}$ and make the identification $C^{\infty}(S^1,\mathfrak{g}) \cong C^{\infty}(S^1,\mathfrak{g}_0) \oplus C^{\infty}(S^1,\mathfrak{t})$ then the operator $\partial/\partial t + \mathrm{ad}(B_0) : C^{\infty}(S^1,\mathfrak{g}) \to C^{\infty}(S^1,\mathfrak{g})$ will leave the two subspaces $C^{\infty}(S^1,\mathfrak{g}_0)$ and $C^{\infty}(S^1,\mathfrak{t})$ invariant.

From the fact that B_0 is in $\mathfrak{t}'_{\mathrm{reg}}$ it follows that for all complex roots α of $\mathfrak{g}$ w.r.t. $\mathfrak{t}$ and all $k \in \mathbb{Z}$ one has $2\pi i k + \alpha(B_0) \neq 0$ (cf. [15], 21.8.4.2). This implies that for every $k \in \mathbb{Z}$ the mapping $(2\pi i k \cdot \mathrm{id}_{\mathfrak{g}_0} + \mathrm{ad}(B_0)_{|\mathfrak{g}_0})\underline{\otimes}\mathrm{id}_{\mathbb{C}} : \mathfrak{g}_0 \otimes \mathbb{C} \to \mathfrak{g}_0 \otimes \mathbb{C}$ is injective and therefore also bijective. So by expanding each $f \in C^{\infty}(S^1,\mathfrak{g}_0 \otimes \mathbb{C})$ in a Fourier series we see that the operator $(\partial/\partial t + \mathrm{ad}(B_0))\underline{\otimes}\mathrm{id}_{\mathbb{C}} : C^{\infty}(S^1,\mathfrak{g}_0 \otimes \mathbb{C}) \to C^{\infty}(S^1,\mathfrak{g}_0 \otimes \mathbb{C})$ and therefore also the operator $\partial/\partial t + \mathrm{ad}(B_0) : C^{\infty}(S^1,\mathfrak{g}_0) \to C^{\infty}(S^1,\mathfrak{g}_0)$ is bijective.

On the other hand the constant functions on S^1 taking values in $\mathfrak{t}$ are in the kernel but not in the image of the operator $\partial/\partial t + \mathrm{ad}(B_0) : C^{\infty}(S^1,\mathfrak{t}) \to C^{\infty}(S^1,\mathfrak{t})$ so this operator is neither injective nor surjective. Let us therefore consider the extension $\bar{C}^{\infty}(S^1,\mathfrak{t})$ of $C^{\infty}(S^1,\mathfrak{t})$ consisting of all those $\mathfrak{t}$-valued functions f on S^1 which are C^{∞} when considered as functions on the semi-open interval $[0,1)$ and which have the additional property that the derivative f' is C^{∞} when considered as a function on S^1 again. More precisely, $\bar{C}^{\infty}(S^1,\mathfrak{t})$ consists of those $\mathfrak{t}$-valued functions f on S^1 such that $f \circ i_{S^1} : [0,1) \to S^1$ is C^{∞} and, additionally, $(\partial/\partial t) f := (f \circ i_{S^1})' \circ i_{S^1}^{-1}$ is an element of $C^{\infty}(S^1,\mathfrak{t})$ (here $(f \circ i_{S^1})'(0)$ is the obvious one-sided derivative in the point 0). It is not difficult to derive the following explicit formula for $\bar{C}^{\infty}(S^1,\mathfrak{t})$, which will be helpful below:

$$\bar{C}^{\infty}(S^1,\mathfrak{t}) = C^{\infty}(S^1,\mathfrak{t}) \oplus \{D_0 \cdot i_{S^1}^{-1}(\cdot) \mid D_0 \in \mathfrak{t}\} \tag{8.1}$$

The operator $\partial/\partial t : C^{\infty}(S^1,\mathfrak{t}) \to C^{\infty}(S^1,\mathfrak{t})$ can be extended in an obvious way to an operator $\bar{C}^{\infty}(S^1,\mathfrak{t}) \to C^{\infty}(S^1,\mathfrak{t})$, which will also be denoted by $\partial/\partial t$. This operator is surjective but neither injective nor anti-symmetric w.r.t. the inner product of $L^2_{\mathfrak{t}}(S^1, dt)$. However, there is a unique subspace $\tilde{C}^{\infty}(S^1,\mathfrak{t})$ of $\bar{C}^{\infty}(S^1,\mathfrak{t})$ such that the restriction $\partial/\partial t : \tilde{C}^{\infty}(S^1,\mathfrak{t}) \to C^{\infty}(S^1,\mathfrak{t})$ is both bijective and anti-symmetric w.r.t. the inner product of $L^2_{\mathfrak{t}}(S^1, dt)$. Using (8.1), it is easy to see that this subspace is given by

$$\begin{aligned}\tilde{C}^{\infty}(S^1,\mathfrak{t}) &= \hat{C}^{\infty}(S^1,\mathfrak{t}) \oplus \left\{ D_0 \cdot \left(i_{S^1}^{-1}(\cdot) - \frac{1}{2} \right) \mid D_0 \in \mathfrak{t} \right\} \\ &= \{ f \in \bar{C}^{\infty}(S^1,\mathfrak{t}) \mid \lim_{u \downarrow 0} f(i_{S^1}(u)) + \lim_{u \uparrow 1} f(i_{S^1}(u)) = 0 \}\end{aligned} \tag{8.2}$$

where $\hat{C}^{\infty}(S^1,\mathfrak{t}) := \{ f \in C^{\infty}(S^1,\mathfrak{t}) \mid f(t_0) = 0 \}$.

As the operator $\partial/\partial t : \tilde{C}^\infty(S^1, \mathfrak{t}) \to C^\infty(S^1, \mathfrak{t})$, which clearly coincides with $\partial/\partial t + \mathrm{ad}(B_0) : \tilde{C}^\infty(S^1, \mathfrak{t}) \to C^\infty(S^1, \mathfrak{t})$, is bijective we also obtain a bijective operator $\partial/\partial t + \mathrm{ad}(B_0) : \tilde{C}^\infty(S^1, \mathfrak{g}) \to C^\infty(S^1, \mathfrak{g})$ where we have set $\tilde{C}^\infty(S^1, \mathfrak{g}) := C^\infty(S^1, \mathfrak{g}_0) \oplus \tilde{C}^\infty(S^1, \mathfrak{t})$. Moreover, this operator is anti-symmetric w.r.t. the inner product of $L^2_{\mathfrak{g}}(S^1, \mathrm{d}t)$. The inverse of this operator will be denoted by $(\partial/\partial t + \mathrm{ad}(B_0))^{-1}$. It has the following properties:

(O1) When restricted onto $C^\infty(S^1, \mathfrak{t})$ the operator $(\partial/\partial t + \mathrm{ad}(B_0))^{-1}$ coincides with $(\partial/\partial t)^{-1} : C^\infty(S^1, \mathfrak{t}) \to \tilde{C}^\infty(S^1, \mathfrak{t})$. It is given by

$$
\begin{aligned}
&((\partial/\partial t + \mathrm{ad}(B_0))^{-1} f)(i_{S^1}(u)) \\
&\quad = ((\partial/\partial t)^{-1} f)(i_{S^1}(u)) = \frac{1}{2}\left[\int_0^u f(i_{S^1}(s))\,\mathrm{d}s - \int_u^1 f(i_{S^1}(s))\,\mathrm{d}s\right]
\end{aligned}
\tag{8.3}
$$

for all $f \in C^\infty(S^1, \mathfrak{t}), u \in [0, 1)$. In particular, $(\partial/\partial t + \mathrm{ad}(B_0))^{-1}$ maps to the constant function on S^1 taking only the value $D_0 \in \mathfrak{t}$ to the function $S^1 \ni t \mapsto (i_{S^1}^{-1}(t) - 1/2) \cdot D_0 \in \mathfrak{t}$.

(O2) $(\partial/\partial t + \mathrm{ad}(B_0))^{-1} : C^\infty(S^1, \mathfrak{g}) \to \tilde{C}^\infty(S^1, \mathfrak{g}) \subset L^2_{\mathfrak{g}}(S^1, \mathrm{d}t)$ is anti-symmetric w.r.t. the scalar product of $L^2_{\mathfrak{g}}(S^1, \mathrm{d}t)$.

(O3) Let $\|\cdot\|_\infty$ be the sup-norm on $\overline{C}^\infty(S^1, \mathfrak{g})$ (w.r.t. $|\cdot|_{\mathfrak{g}}$). Eq. (8.3) shows that $(\partial/\partial t)^{-1} : C^\infty(S^1, \mathfrak{t}) \to \tilde{C}^\infty(S^1, \mathfrak{t})$ is $\|\cdot\|_\infty$-continuous. In Remark 8.1 below, we show that the same is true for the restriction of $(\partial/\partial t + \mathrm{ad}(B_0))^{-1}$ onto $C^\infty(S^1, \mathfrak{g}_0)$. From this it follows that also the original operator $(\partial/\partial t + \mathrm{ad}(B_0))^{-1} : C^\infty(S^1, \mathfrak{g}) \to \tilde{C}^\infty(S^1, \mathfrak{g})$ is $\|\cdot\|_\infty$-continuous.

Remark 8.1. The operator $(\partial/\partial t + \mathrm{ad}(B_0))^{-1} : C^\infty(S^1, \mathfrak{g}_0) \to C^\infty(S^1, \mathfrak{g}_0)$ is $\|\cdot\|_\infty$-continuous. This can be seen as follows:

Let $f \in C^\infty(S^1, \mathfrak{g}_0)$ and set $g := (\partial/\partial t + \mathrm{ad}(B_0))^{-1} \cdot f \in C^\infty(S^1, \mathfrak{g}_0)$. Using the method of "variation of constants" one can easily derive the following explicit formula for g:

$$
\begin{aligned}
\forall\, u \in [0, 1] : \quad g(i_{S^1}(u)) = &\int_0^u \exp((s-u)\cdot \mathrm{ad}(B_0)) f(i_{S^1}(s))\,\mathrm{d}s \\
&+ g(i_{S^1}(0)) \exp(-u \cdot \mathrm{ad}(B_0))
\end{aligned}
\tag{8.4}
$$

Because of $g(i_{S^1}(1)) = g(t_0) = g(i_{S^1}(0))$ we obtain for the special case $u = 1$:

$$
(\exp(\mathrm{ad}(B_0)_{|\mathfrak{g}_0}) - \mathrm{id}_{\mathfrak{g}_0}) \cdot g(t_0) = \int_0^1 \exp(s \cdot \mathrm{ad}(B_0)) f(i_{S^1}(s))\,\mathrm{d}s
$$

From the assumption that $B_0 \in \mathfrak{t}'_{\mathrm{reg}}$ it follows that for every complex root α we have $\alpha(B_0) \notin 2\pi i\mathbb{Z}$ so 1 is not an eigenvalue of $\exp(\mathrm{ad}(B_0)_{|\mathfrak{g}_0})$ which means that $\exp(\mathrm{ad}(B_0)_{|\mathfrak{g}_0}) - \mathrm{id}_{\mathfrak{g}_0} \in$

$\mathrm{End}(\mathfrak{g}_0)$ is invertible. So we obtain

$$|g(t_0)|_{\mathfrak{g}} = \left| \left(\exp(\mathrm{ad}(B_0)_{|\mathfrak{g}_0}) - \mathrm{id}_{\mathfrak{g}_0}\right)^{-1} \cdot \int_0^1 \exp(s \cdot \mathrm{ad}(B_0)) f(i_{S^1}(s))\, \mathrm{d}s \right|_{\mathfrak{g}}$$

Without loss of generality we can assume that $|g(t_0)|_{\mathfrak{g}} = \|g\|_\infty$ (note that instead of the mapping i_{S^1} in 8.4 we can use any other mapping which is obtained from i_{S^1} by a translation). From this we obtain immediately $\|(\partial/\partial t + \mathrm{ad}(B_0))^{-1} \cdot f\|_\infty = \|g\|_\infty \leq M\|f\|_\infty$ with a suitable constant $M > 0$ independent of *f*.

8.2. A rigorous implementation of $\int \cdots \mathrm{d}\hat{\mu}_B^{\perp}$

Let us fix $B \in C^\infty(\Sigma, \mathfrak{t}'_{\mathrm{reg}})$. Let $\bar{C}^\infty(S^1, \mathcal{A}_\Sigma)$ be defined in an analogous way as the space $\bar{C}^\infty(S^1, \mathfrak{g})$ above, i.e., we set $\bar{C}^\infty(S^1, \mathcal{A}_\Sigma) := C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}_0}) \oplus \bar{C}^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}})$, where $\bar{C}^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}}) := C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}}) \oplus \{D \cdot i_{S^1}^{-1}(\cdot) \mid D \in \mathcal{A}_{\Sigma,\mathfrak{t}}\}$, cf. (8.1) above. Moreover, we set $\tilde{C}^\infty(S^1, \mathcal{A}_\Sigma) := C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}_0}) \oplus \tilde{C}^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}})$, where $\tilde{C}^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}}) := \hat{C}^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}}) \oplus \{D \cdot (i_{S^1}^{-1}(\cdot) - 1/2) \mid D \in \mathcal{A}_{\Sigma,\mathfrak{t}}\}$ with $\hat{C}^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}}) := \{A^\perp \in C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}}) \mid A^\perp(t_0) = 0\}$, cf. (8.2) above. $\partial/\partial t$ will denote the obvious operator $\bar{C}^\infty(S^1, \mathcal{A}_\Sigma) \to C^\infty(S^1, \mathcal{A}_\Sigma)$ and $VF(\Sigma)$ the space of smooth vector fields on Σ.

Definition 8.1. By $\langle \cdot, \cdot \rangle_M$ we denote the bilinear form on $\bar{C}^\infty(S^1, \mathcal{A}_\Sigma)$ given by $\langle A, A' \rangle_M = \int_{S^1} \langle A(t), A'(t) \rangle_\Sigma \, \mathrm{d}t$ for all $A, A' \in \bar{C}^\infty(S^1, \mathcal{A}_\Sigma)$.

Proposition 8.1.

(i) *The operator* $\partial/\partial t + \mathrm{ad}(B) : \tilde{C}^\infty(S^1, \mathcal{A}_\Sigma) \to C^\infty(S^1, \mathcal{A}_\Sigma)$ *is bijective and its inverse* $(\partial/\partial t + \mathrm{ad}(B))^{-1} : C^\infty(S^1, \mathcal{A}_\Sigma) \to \tilde{C}^\infty(S^1, \mathcal{A}_\Sigma)$ *is given by*

$$((\partial/\partial t + \mathrm{ad}(B))^{-1} \cdot A^\perp)(\cdot)(X_\sigma) = (\partial/\partial t + \mathrm{ad}(B(\sigma)))^{-1} \cdot (A^\perp(\cdot)(X_\sigma)) \tag{8.5}$$

for all $A^\perp \in C^\infty(S^1, \mathcal{A}_\Sigma)$, $X \in VF(\Sigma)$, $\sigma \in \Sigma$ *where* $(\partial/\partial t + \mathrm{ad}(B(\sigma)))^{-1}$ *is as in Section 8.1.*

(ii) *The operators* $\partial/\partial t + \mathrm{ad}(B) : \tilde{C}^\infty(S^1, \mathcal{A}_\Sigma) \to C^\infty(S^1, \mathcal{A}_\Sigma)$ *and* $(\partial/\partial t + \mathrm{ad}(B))^{-1} : C^\infty(S^1, \mathcal{A}_\Sigma) \to \tilde{C}^\infty(S^1, \mathcal{A}_\Sigma)$ *are anti-symmetric w.r.t.* $\langle \cdot, \cdot \rangle_M$.

Proof. Part (i): That $\partial/\partial t + \mathrm{ad}(B)$ is injective follows immediately from the fact that for each $\sigma \in \Sigma$ the mapping $\partial/\partial t + \mathrm{ad}(B(\sigma)) : \tilde{C}^\infty(S^1, \mathfrak{g}) \to C^\infty(S^1, \mathfrak{g})$ is injective. Moreover, as each $\partial/\partial t + \mathrm{ad}(B(\sigma)) : \tilde{C}^\infty(S^1, \mathfrak{g}) \to C^\infty(S^1, \mathfrak{g})$ is also surjective the right-hand side of Eq. (8.5) is well-defined for all $A^\perp \in C^\infty(S^1, \mathcal{A}_\Sigma)$, $X \in VF(\Sigma)$, and $\sigma \in \Sigma$. Thus, there is a unique function $A_0^\perp : S^1 \to \mathcal{A}_\Sigma$ such that $A_0^\perp(\cdot)(X_\sigma) = (\partial/\partial t + \mathrm{ad}(B(\sigma)))^{-1} \cdot (A^\perp(\cdot)(X_\sigma))$ for all $X \in VF(\Sigma)$ and $\sigma \in \Sigma$. One can show that $A_0^\perp$ is indeed an element of $\tilde{C}^\infty(S^1, \mathcal{A}_\Sigma)$. Finally, it is a trivial matter to check that $(\partial/\partial t + \mathrm{ad}(B)) \cdot A_0^\perp = A^\perp$, which implies that $\partial/\partial t + \mathrm{ad}(B)$ is surjective and that (8.5) holds.

Part (ii): Making use of the fact that the maps $\partial/\partial t + \mathrm{ad}(B(\sigma)) : \tilde{C}^\infty(S^1, \mathfrak{g}) \to C^\infty(S^1, \mathfrak{g})$, $\sigma \in \Sigma$, are anti-symmetric w.r.t. the inner product of $L^2_{\mathfrak{g}}(S^1, \mathrm{d}t)$ the anti-symmetry of $\partial/\partial t + \mathrm{ad}(B)$ follows from a short computation which involves the definitions of $\langle\cdot,\cdot\rangle_M$ and $\langle\cdot,\cdot\rangle_\Sigma$. This implies immediately that also $(\partial/\partial t + \mathrm{ad}(B))^{-1}$ is anti-symmetric w.r.t. $\langle\cdot,\cdot\rangle_M$. □

Let us set

$$m(B) := (\partial/\partial t + \mathrm{ad}(B))^{-1} \cdot \mathrm{d}B = (\partial/\partial t)^{-1} \cdot \mathrm{d}B \in \tilde{C}^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}}) \tag{8.6}$$

From Proposition 8.1 and (O1) above it follows that

$$m(B)(t) = (i^{-1}_{S^1}(t) - \tfrac{1}{2}) \cdot \mathrm{d}B \quad \text{for all } t \in S^1 \tag{8.7}$$

Let us recall the definition of the space $\hat{\mathcal{A}}^\perp$ in (6.5a) (which coincides with the space $\hat{\mathcal{A}}^\perp$ in Section 6.1 if G is Abelian). Clearly,

$$\begin{aligned}\tilde{C}^\infty(S^1, \mathcal{A}_\Sigma) &= C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}_0}) \oplus \hat{C}^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{t}}) \oplus \{D \cdot (i^{-1}_{S^1}(\cdot) - 1/2) \mid D \in \mathcal{A}_{\Sigma,\mathfrak{t}}\}\\ &= \hat{\mathcal{A}}^\perp \oplus \{D \cdot (i^{-1}_{S^1}(\cdot) - \tfrac{1}{2}) \mid D \in \mathcal{A}_{\Sigma,\mathfrak{t}}\}\end{aligned}$$

so $\hat{A}^\perp - m(B) \in \tilde{C}^\infty(S^1, \mathcal{A}_\Sigma)$ for every $\hat{A}^\perp \in \hat{\mathcal{A}}^\perp$.

Consequently, by taking into account Proposition 5.2, Eq. (8.6), the anti-symmetry of $\partial/\partial t + \mathrm{ad}(B)$ on $\tilde{C}^\infty(S^1, \mathcal{A}_\Sigma)$ w.r.t. the anti-symmetric bilinear form $\langle\cdot,\cdot\rangle_M$, and the two relations $\langle m(B), \mathrm{d}B\rangle_M = 0$ and $\mathrm{ad}(B)\mathrm{d}B = 0$ we obtain for $\hat{A}^\perp \in \hat{\mathcal{A}}^\perp$

$$S_{CS}(\hat{A}^\perp + B\,\mathrm{d}t) = -\frac{k}{4\pi}\langle \hat{A}^\perp - m(B), (\partial/\partial t + \mathrm{ad}(B)) \cdot (\hat{A}^\perp - m(B))\rangle_M \tag{8.8}$$

Remark 8.2. Eq. (8.8) will not hold if we replace $\hat{A}^\perp \in \hat{\mathcal{A}}^\perp$ by a general element $A^\perp \in \mathcal{A}^\perp$. Instead, one then obtains

$$\begin{aligned}&S_{CS}(A^\perp + B\,\mathrm{d}t)\\ &\quad = -\frac{k}{4\pi}[\langle A^\perp - m(B), (\partial/\partial t + \mathrm{ad}(B)) \cdot (A^\perp - m(B))\rangle_M + \langle A^\perp(t_0), \mathrm{d}B\rangle_\Sigma]\end{aligned}$$

So if we had not introduced the decomposition (6.5) above we would now be lead to it more or less automatically in order to deal appropriately with the singular term $\langle A^\perp(t_0), \mathrm{d}B\rangle_\Sigma$.

At first look Eq. (8.8) seems to suggest that the heuristic measure (6.2) on $\hat{\mathcal{A}}^\perp$ is "Gaussian" with "mean" $m(B)$ and "covariance operator" $-(2\pi i/k)(\partial/\partial t + \mathrm{ad}(B))^{-1}$ w.r.t. $\langle\cdot,\cdot\rangle_M$. However, as $\langle\cdot,\cdot\rangle_M$ is not a scalar product one has to be more careful. In order to obtain a genuine scalar product let us now fix an auxiliary Riemannian metric $\mathbf{g}$ on Σ, which we have so far avoided. Then we obtain a scalar product $\langle\langle\cdot,\cdot\rangle\rangle_{\hat{\mathcal{A}}^\perp}$ on $\hat{\mathcal{A}}^\perp$ given by $\langle\langle \hat{A}^\perp_1, \hat{A}^\perp_2\rangle\rangle_{\hat{\mathcal{A}}^\perp} = \int_{S^1}(\int_\Sigma (\hat{A}^\perp_1(t), \hat{A}^\perp_2(t))_{\mathbf{g}}\, \mathrm{d}\mu_{\mathbf{g}})\,\mathrm{d}t$ for all $\hat{A}^\perp_1, \hat{A}^\perp_2 \in \hat{\mathcal{A}}^\perp$ where $\mu_{\mathbf{g}}$ denotes the Riemannian volume measure on Σ associated to $\mathbf{g}$ and where $(\cdot,\cdot)_{\mathbf{g}}$ is the fibre metric on the bundle $\mathrm{Hom}(T\Sigma, \mathfrak{g}) \cong T\Sigma^* \otimes \mathfrak{g}$ induced by $\mathbf{g}$ and $(\cdot,\cdot)_{\mathfrak{g}}$. Note that $\langle\langle\cdot,\cdot\rangle\rangle_{\hat{\mathcal{A}}^\perp}$ is just the restriction onto $\hat{\mathcal{A}}^\perp$ of the standard scalar product $\langle\langle\cdot,\cdot\rangle\rangle_{L^2}$ of the Hilbert space L^2-$\Gamma(\mathrm{Hom}(TM, \mathfrak{g}), \mu_{\mathbf{g}} \otimes \mathrm{d}t)$ of L^2-sections of the bundle $\mathrm{Hom}(TM, \mathfrak{g})$ w.r.t. the measure $\mu_{\mathbf{g}} \otimes \mathrm{d}t$ on M. Here we have equipped $\mathrm{Hom}(TM, \mathfrak{g}) \cong TM^* \otimes \mathfrak{g}$ with the obvious fibre metric. In the sequel, we will identify the completion $(\mathcal{H}, \langle\langle\cdot,\cdot\rangle\rangle_{\mathcal{H}})$ of the Pre-Hilbert space $(\hat{\mathcal{A}}^{\perp}, \langle\langle\cdot,\cdot\rangle\rangle_{\hat{\mathcal{A}}^{\perp}})$ with the obvious closed subspace of L^2-$\Gamma(\mathrm{Hom}(TM,\mathfrak{g}), \mu_{\boldsymbol{g}} \otimes \mathrm{d}t)$.

Remark 8.3.

(i) After the identifications mentioned above we have the inclusion $\hat{\mathcal{A}}^{\perp} \subset \mathcal{A}^{\perp} \subset \bar{C}^{\infty}(S^1, \mathcal{A}_{\Sigma}) \subset \mathcal{H} \subset L^2-\Gamma(\mathrm{Hom}(TM,\mathfrak{g}), \mu_{\boldsymbol{g}} \otimes \mathrm{d}t)$. In particular, we have $\mathcal{A}^{\perp} \subset \mathcal{H} \cap \mathcal{A}$.

(ii) The Hodge star operator $\star$ on $\Omega^1(M) \otimes \mathfrak{g} \cong \mathcal{A}$ (where we have equipped $M = \Sigma \times S^1$ with the product of $\boldsymbol{g}$ with the standard Riemannian metric on S^1) leaves the subspace $\mathcal{A}^{\perp}$ of $\mathcal{A}$ invariant. Thus, we obtain a linear operator on $\mathcal{A}^{\perp} \to \mathcal{A}^{\perp}$ which can be shown to be bijective and $\|\cdot\|_{\mathcal{H}}$-bounded. Its continuous extension to a linear isomorphism of $\mathcal{H}$ will also be denoted by $\star$.

We can now rewrite (8.8) in the form

$$S_{CS}(\hat{A}^{\perp} + B\,\mathrm{d}t) = -\frac{k}{4\pi}\langle\langle \hat{A}^{\perp} - m(B), (\star \circ (\partial/\partial t + \mathrm{ad}(B))) \cdot (\hat{A}^{\perp} - m(B))\rangle\rangle_{\mathcal{H}} \tag{8.9}$$

where $\star : \mathcal{H} \to \mathcal{H}$ is as in Remark 8.3(ii). Eq. (8.9) suggests that the heuristic measure (6.2) on $\hat{\mathcal{A}}^{\perp}$ is "Gaussian" with "mean" $m(B)$ and "covariance operator" $C(B) := -(2\pi i/k)(\partial/\partial t + \mathrm{ad}(B))^{-1} \circ \star^{-1}$ w.r.t. $\langle\langle\cdot,\cdot\rangle\rangle_{\mathcal{H}}$ and that $\hat{Z}(B) \sim |\det(\partial/\partial t + \mathrm{ad}(B))|^{-1/2}$. Informally, the Fourier transformation $\mathcal{F}\hat{\mu}^{\perp}_B$ of $\hat{\mu}^{\perp}_B$ is given by

$$\begin{aligned} \mathcal{F}\hat{\mu}^{\perp}_B(\hat{A}^{\perp}_0) &= \int \exp(i\langle\langle \hat{A}^{\perp}_0, \hat{A}^{\perp}\rangle\rangle_{\mathcal{H}})\,\mathrm{d}\hat{\mu}^{\perp}_B(\hat{A}^{\perp}) \\ &= \exp(i\langle\langle \hat{A}^{\perp}_0, m(B)\rangle\rangle_{\mathcal{H}}) \\ &\quad \times \exp(i\pi\lambda\langle\langle \hat{A}^{\perp}_0, ((\partial/\partial t + \mathrm{ad}(B))^{-1} \circ \star^{-1})\hat{A}^{\perp}_0\rangle\rangle_{\mathcal{H}}) \ \ \forall\, \hat{A}^{\perp}_0 \in \hat{\mathcal{A}}^{\perp} \end{aligned} \tag{8.10}$$

Let us now explain how, with the help of (8.10), it is possible to make rigorous sense of the integral functional $\int \cdots \mathrm{d}\hat{\mu}^{\perp}_B$ as a generalized distribution on a suitable extension $\overline{\hat{\mathcal{A}}^{\perp}}$ of the space $\hat{\mathcal{A}}^{\perp}$.

Let $\Delta : \Omega^{\cdot}(M) \to \Omega^{\cdot}(M)$ denote the Hodge-Laplace operator on $M = \Sigma \times S^1$. It is easy to show that Δ leaves $\mathcal{A}^{\perp}$ invariant so $\Delta_{|\mathcal{A}^{\perp}}$ can be considered as an operator on $\mathcal{H}$ with dense domain $\mathcal{A}^{\perp}$. One can prove that the operator $\mathcal{K} := (-\Delta_{|\mathcal{A}^{\perp}} + 1)^{-1}$ is a self-adjoint Hilbert–Schmidt operator on $\mathcal{H}$. So we can apply the machinery of Section 7 to the pair $(\mathcal{H}, \mathcal{K})$ obtaining the spaces $\mathcal{N}$, $\mathcal{N}^*$, $\mathcal{P}(\mathcal{N})$, $\mathcal{E}(\mathcal{N})$, $(\mathcal{N})$, and $(\mathcal{N})^*$. Using a Sobolev embedding argument one can prove that $\mathcal{N} = \mathcal{A}^{\perp} \cong C^{\infty}(S^1, \mathcal{A}_{\Sigma})$. Consequently, the following definition makes sense:

Definition 8.2. Let $a^{\perp}_B$ denote the linear form on $\mathcal{N}$ given by $a^{\perp}_B(j) = \langle\langle j, m(B)\rangle\rangle_{\mathcal{H}}$ for all $j \in \mathcal{N}$. Let $C_0(B) : \mathcal{N} \to \bar{C}^{\infty}(S^1, \mathcal{A}_{\Sigma}) \subset \mathcal{H}$ denote the linear operator given by $C_0(B) \cdot j =$

$((\partial/\partial t + \mathrm{ad}(B))^{-1} \circ \star^{-1}) \cdot j$ for all $j \in \mathcal{N}$. and let $Q_B^{\perp}$ denote the bilinear form on $\mathcal{N}$ given by

$$Q_B^{\perp}(j_1, j_2) = \langle\langle j_1, C_0(B) j_2 \rangle\rangle_{\mathcal{H}} \overset{(*)}{=} -\langle j_1, (\partial/\partial t + \mathrm{ad}(B))^{-1} j_2 \rangle_M \tag{8.11}$$

for all $j_1, j_2 \in \mathcal{N}$. Here step $(*)$ follows because $\star \cdot (\partial/\partial t + \mathrm{ad}(B))^{-1} = (\partial/\partial t + \mathrm{ad}(B))^{-1} \cdot \star$ and $\star^{-1} = -\star$. From (8.11) and the fact that $(\partial/\partial t + \mathrm{ad}(B))^{-1} : \mathcal{N} \to \bar{C}^{\infty}(S^1, \mathcal{A}_{\Sigma})$ is anti-symmetric w.r.t. $\langle \cdot, \cdot \rangle_M$ (cf. Proposition 8.1) and the bilinear form $\langle \cdot, \cdot \rangle_M$ is anti-symmetric itself it follows immediately that the bilinear form $Q_B^{\perp}$ is symmetric. Hence, also the operator $C_0(B)$ on $\mathcal{H}$ is symmetric.

Remark 8.4. We observe that while the linear form $a_B^{\perp}$ depends on the special choice of the Riemannian metric $\boldsymbol{g}$ on Σ the bilinear form $Q_B^{\perp}$ does not.

As the standard topology on $\mathcal{N}$ is finer than the topology induced by $\|\cdot\|_{\mathcal{H}}$ it follows immediately that the linear form $a_B^{\perp}$ is continuous. One can show that the densely defined operator $C_0(B)$ on $\mathcal{H}$ is bounded, from which it follows immediately that also $Q_B^{\perp}$ is continuous. However, instead of giving the details of the last argument we prefer to give a direct proof for the the continuity of $Q_B^{\perp}$.

Proposition 8.2. *The bilinear form $Q_B^{\perp}$ on $\mathcal{N}$ is continuous.*

Proof. Let $\|\cdot\|_{\infty}$ denote the norm on $\bar{C}^{\infty}(S^1, \mathcal{A}_{\Sigma})$ which is given by $\|A^{\perp}\|_{\infty} = \sup_{t \in S^1} \sup_{\sigma \in \Sigma} \sup_{X_\sigma \in T_\sigma \Sigma, \|X_\sigma\|_{\boldsymbol{g}} \leq 1} |A^{\perp}(t)(X_\sigma)|_{\mathfrak{g}}$, $A^{\perp} \in \mathcal{A}^{\perp}$. It is easy to see that $\langle \cdot, \cdot \rangle_M$ is a $\|\cdot\|_{\infty}$-continuous bilinear form on $\bar{C}^{\infty}(S^1, \mathcal{A}_{\Sigma})$. Moreover, we know from (O3) in Section 8.1 that for each $\sigma \in \Sigma$ the operator $(\partial/\partial t + \mathrm{ad}(B(\sigma)))^{-1} : C^{\infty}(S^1, \mathfrak{g}) \to \tilde{C}^{\infty}(S^1, \mathfrak{g})$ considered as a densely defined operator on $\bar{C}^{\infty}(S^1, \mathfrak{g})$ is continuous w.r.t. $\|\cdot\|_{\infty}$. Let $\|\cdot\|$ denote the operator norm of $(\bar{C}^{\infty}(S^1, \mathfrak{g}), \|\cdot\|_{\infty})$. Then it is not difficult to see that $M_B := \sup_{\sigma \in \Sigma} \|(\partial/\partial t + \mathrm{ad}(B(\sigma)))^{-1}\| < \infty$. Eq. (8.5) then implies $\|(\partial/\partial t + \mathrm{ad}(B))^{-1} \cdot A^{\perp}\|_{\infty} \leq M_B \|A^{\perp}\|_{\infty}$ for every $A^{\perp} \in C^{\infty}(S^1, \mathcal{A}_{\Sigma})$. The assertion now follows from Eq. (8.11). $\square$

Taking into account Theorem 7.1, Remark 7.1, Proposition 8.2, and Eq. (8.10) above we now arrive at the following rigorous realization of the informal integral functional $\int \cdots \mathrm{d}\hat{\mu}_B^{\perp}$ as a generalized distribution $\Phi_B^{\perp}$ on $\mathcal{N}^* =: \overline{\hat{\mathcal{A}}^{\perp}}$.

Definition 8.3. The Gaussian element of $(\mathcal{N})^*$ corresponding to $(a_B^{\perp}, -2\pi\lambda i Q_B^{\perp})$ will be denoted by $\Phi_B^{\perp}$.

9. Regularization techniques

9.1. Admissible links

Let π_{Σ} (resp. π_{S^1}) denote the canonical projection $\Sigma \times S^1 \to \Sigma$ (resp. $\Sigma \times S^1 \to S^1$). For every curve c in $\Sigma \times S^1$, i.e., every smooth function $[0, 1] \to \Sigma \times S^1$, we set $c_{\Sigma} := \pi_{\Sigma} \circ c$ and $c_{S^1} := \pi_{S^1} \circ c$.

Let $C = (c_1, \ldots, c_r)$, $r \in \mathbb{N}$, be an r-tuple of curves in $\Sigma \times S^1$. A double point of C (resp. a triple point of C) is an element p of Σ with the property that the intersection of

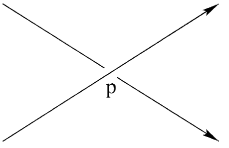

Fig. 1. $\epsilon(p) = -1$.

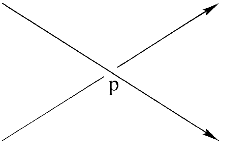

Fig. 2. $\epsilon(p) = 1$.

$\pi_{\Sigma}^{-1}(\{p\})$ with the union of the arcs of the curves $c_1, \ldots, c_r$ contains at least two (resp. three) elements. We will denote the set of double points of C by $DP(C)$.

In the sequel we will identify every "loop" l in $\Sigma \times S^1$ in the sense of Section 5.1 with the curve $l \circ i_{S^1}$ and every "link" in $\Sigma \times S^1$ with the obvious finite tuple of curves.

Definition 9.1. A link $L = (l^1, \ldots, l^n)$, $n \in \mathbb{N}$, in $\Sigma \times S^1$ is called admissible iff the following conditions are fulfilled:

(A1) There are only finitely many double and no triple points of L.
(A2) For all $i, j \leq n$ and all $\bar{v}, \bar{u} \in [0, 1]$ such that $l^i_{\Sigma}(\bar{v}) = l^j_{\Sigma}(\bar{u})$ the two tangent vectors $(l^i_{\Sigma})'(\bar{v})$ and $(l^j_{\Sigma})'(\bar{u})$ are not parallel to each other and. In particular, both vectors are non-zero.
(A3) $(l^i_{S^1})^{-1}(\{t_0\})$ is finite for all $i \leq n$.
(A4) For all $i \leq n$ and $u \in [0, 1]$ such that $l^i_{\Sigma}(u) \in DP(L)$ we have $l^i_{S^1}(u) \neq t_0$

For every admissible two-component link $(l, \tilde{l})$ in $\Sigma \times S^1$ we set

$$\mathrm{LK}^*(l, \tilde{l}) := \sum_{p \in DP(l,\tilde{l}) \backslash (DP(l) \cup DP(\tilde{l}))} \frac{1}{2} \epsilon(p),$$

where for every $p \in DP(l, \tilde{l})$ the number $\epsilon(p) \in \{-1, 1\}$ is given by $\epsilon(p) := -1$ in the situation of Fig. 1 and $\epsilon(p) := 1$ in the situation of Fig. 2. Note that for a sufficiently small neighborhood U of p property (A4) of the link $(l, \tilde{l})$ implies that the parts of the arcs of l and $\tilde{l}$ which are contained in $U \times S^1$ can be considered to be subsets of $U \times (S^1 \backslash \{t_0\}) \cong U \times \mathbb{R}$. (For a rigorous definition of $\epsilon(p)$ see Eq. (10.14) below).

Remark 9.1. If $(l_{S^1})^{-1}(\{t_0\})$ and $(\tilde{l}_{S^1})^{-1}(\{t_0\})$ are empty then $\mathrm{LK}^*(l, \tilde{l})$ coincides with the linking number $\mathrm{LK}(l, \tilde{l})$ of l and $\tilde{l}$.

9.2. Loop smearing

In Section 8, we succeeded in defining the integral functional $\int \cdots d\hat{\mu}^{\perp}_B$ as a (generalized) distribution $\Phi^{\perp}_B$ on $\overline{\hat{\mathcal{A}}^{\perp}} = \mathcal{N}^*$. For sufficiently regular links L and fixed $B \in C^{\infty}(\Sigma, \mathfrak{t}'_{\text{reg}})$, $A^{\perp}_c \in \mathcal{A}^{\perp}_c$ we would now like to use $\Phi^{\perp}_B$ to make sense of $\int WLF(L)(\hat{A}^{\perp} + A^{\perp}_c + B\,dt)\, d\hat{\mu}^{\perp}_B(\hat{A}^{\perp})$. If $\hat{A}^{\perp} \in \hat{\mathcal{A}}^{\perp}$ then $WLF(L)(\hat{A}^{\perp} + A^{\perp}_c + B\,dt)$ is obtained by solving an ODE involving the expression $(\hat{A}^{\perp} + A^{\perp}_c + B\,dt)(l'(u))$, $u \in [0, 1]$. Unfortunately, for a general element $\hat{A}^{\perp} \in \overline{\hat{\mathcal{A}}^{\perp}} = \mathcal{N}^*$ the expression $\hat{A}^{\perp}(l'(u))$ makes no sense.

This problem can be solved by "smearing" the loops considered: We will replace the expression $\hat{A}^{\perp}(l'(u))$ by another expression (the $\langle \hat{A}^{\perp}, h^{l^{\epsilon}}(u)\rangle$ below) which is obtained using a "smearing" of l, i.e., a family $(l^{\epsilon}(u))_{u\in[0,1],\epsilon>0}$ of test functions such that for all $u \in [0, 1]$, $\epsilon > 0$ the support of $l^{\epsilon}(u)$ is contained in an ϵ-neighborhood of $l(u)$. Later we will let ϵ go to zero. More precisely, let us fix two families $(\eta^{\epsilon}_{t'})_{\epsilon>0,t'\in S^1}$ and $(\psi^{\epsilon}_{\sigma'})_{\epsilon>0,\sigma'\in\Sigma}$ such that $\eta^{\epsilon}_{t'} \in C^{\infty}(S^1, \mathbb{R})$, $\eta^{\epsilon}_{t'} \geq 0$, $\int \eta^{\epsilon}_{t'}(t)\, dt = 1$, $\text{supp}(\eta^{\epsilon}_{t'}) \subset B_{\epsilon}(t')$ and $\psi^{\epsilon}_{\sigma'} \in C^{\infty}(\Sigma, \mathbb{R})$, $\psi^{\epsilon}_{\sigma'} \geq 0$, $\int \psi^{\epsilon}_{\sigma'}(\sigma)\, d\mu_{\boldsymbol{g}}(\sigma) = 1$, $\text{supp}(\psi^{\epsilon}_{\sigma'}) \subset B_{\epsilon}(\sigma')$ for all $t' \in S^1$, $\epsilon > 0$ where $B_{\epsilon}(t')$ (resp. $B_{\epsilon}(\sigma')$) is the open ball in S^1 (resp. Σ) around t' (resp. σ') with radius ϵ w.r.t. the standard distance function d_{S^1} on S^1 (resp. the distance function $d_{\boldsymbol{g}}$ on Σ which is induced by $\boldsymbol{g}$). Below (cf. Remark 9.2) we will give some additional conditions which the family $(\psi^{\epsilon}_{\sigma'})_{\epsilon>0,\sigma'\in\Sigma}$ has to fulfill. For every loop l in M and every $u \in [0, 1]$ we define $l^{\epsilon}(u) \in C^{\infty}(M, \mathbb{R})$ by

$$l^{\epsilon}(u)(\sigma, t) := \psi^{\epsilon}_{l_{\Sigma}(u)}(\sigma)\eta^{\epsilon}_{l_{S^1}(u)}(t) \quad \text{for all } \sigma \in \Sigma, t \in S^1 \tag{9.1}$$

Let $C^{\infty}(S^1, VF(\Sigma))$ be defined in an analogous way as $C^{\infty}(S^1, \mathcal{A}_{\Sigma})$ in Definition 5.1. As Σ is compact there is a $\epsilon_0 > 0$ such that for all $\sigma_1, \sigma_2 \in \Sigma$ with $d_{\boldsymbol{g}}(\sigma_1, \sigma_2) \leq \epsilon_0$ there is a unique segment $\gamma(\sigma_1, \sigma_2)$ starting in σ_1 and ending in σ_2 (cf. [27], Chap. 5). For every $u \in [0, 1]$ and $\epsilon < \epsilon_0$ we therefore obtain an element $h^{l^{\epsilon}}(u)$ of $C^{\infty}(S^1, VF(\Sigma))$ which is given by

$$[h^{l^{\epsilon}}(u)(t)](\sigma) = \begin{cases} PT_{\gamma(l_{\Sigma}(u),\sigma)}(l'_{\Sigma}(u))l^{\epsilon}(u)(\sigma, t), & \text{if } \sigma \in B_{\epsilon}(l_{\Sigma}(u)), \\ 0, & \text{if } \sigma \notin B_{\epsilon}(l_{\Sigma}(u)) \end{cases}$$

for all $t \in S^1$ where $PT_{\gamma(l_{\Sigma}(u),\sigma)}$ is the parallel transport operator along $\gamma(l_{\Sigma}(u), \sigma)$ w.r.t. the Levi-Civita connection.

Let $\langle\cdot,\cdot\rangle : C^{\infty}(S^1, \mathcal{A}_{\Sigma}) \times C^{\infty}(S^1, VF(\Sigma)) \to \mathfrak{g}$ denote the bilinear map given by $\langle \hat{A}^{\perp}, h\rangle = \int \left[\int \hat{A}^{\perp}(t)(h(t))\, d\mu_{\boldsymbol{g}}\right] dt$, $\hat{A}^{\perp} \in C^{\infty}(S^1, \mathcal{A}_{\Sigma})$, $h \in C^{\infty}(S^1, VF(\Sigma))$. The expression $\langle \hat{A}^{\perp}, h^{l^{\epsilon}}(u)\rangle$, $\hat{A}^{\perp} \in \hat{\mathcal{A}}^{\perp} \subset C^{\infty}(S^1, \mathcal{A}_{\Sigma})$, can be considered as a "smeared analogue" of $\hat{A}^{\perp}(l'(u))$. In order to find the generalization of $\langle \hat{A}^{\perp}, h^{l^{\epsilon}}(u)\rangle$ for an arbitrary element $\hat{A}^{\perp}$ of $\mathcal{N}^*$ we have to rewrite the expression $\langle \hat{A}^{\perp}, h^{l^{\epsilon}}(u)\rangle$ in terms of the pairing $(\cdot,\cdot)$ between $\mathcal{N}^*$ and $\mathcal{N}$. For this purpose consider the linear isomorphism $\xi : VF(\Sigma) \to \mathcal{A}_{\Sigma,\mathbb{R}}$ given by $\alpha_{\sigma}(j_{\sigma}) = (\alpha_{\sigma}, \xi(j)_{\sigma})_{\boldsymbol{g}}$ for all $\alpha \in \mathcal{A}_{\Sigma,\mathbb{R}}$, $j \in VF(\Sigma)$, $\sigma \in \Sigma$ where $(\cdot,\cdot)_{\boldsymbol{g}}$ denotes the fiber metric on $T\Sigma^*$ which is induced by $\boldsymbol{g}$. For each $a \leq \dim \mathfrak{g}$, $u \in [0, 1]$, $f^{l^{\epsilon}}_a(u)$ will denote the element of $C^{\infty}(S^1, \mathcal{A}_{\Sigma,\mathbb{R}}) \otimes \mathfrak{g} \cong \mathcal{A}^{\perp} = \mathcal{N}$ given by $f^{l^{\epsilon}}_a(u)(t) := \xi(h^{l^{\epsilon}}(u)(t)) \otimes T_a$

for all $t \in S^1$, where $(T_a)_a$ is a fixed ONB of $\mathfrak{g}$. Clearly, $\langle \hat{A}^\perp, h^{l^\epsilon}(u) \rangle = \sum_a T_a(\hat{A}^\perp, f_a^{l^\epsilon}(u))$. As $\sum_a T_a(\hat{A}^\perp, f_{\mathfrak{g}}^{l^\epsilon}(u))$ is defined for arbitrary $\hat{A}^\perp \in \mathcal{N}^*$ we can now introduce a "smeared" holonomy $\mathrm{Hol}(\hat{A}^\perp, l^\epsilon; A_c^\perp, B)$ for $\hat{A}^\perp \in \mathcal{N}^*$, $A_c^\perp \in \mathcal{A}_c^\perp$, $B \in C^\infty(\Sigma, \mathfrak{t}'_{\mathrm{reg}})$, by setting $\mathrm{Hol}(\hat{A}^\perp, l^\epsilon; A_c^\perp, B) := P^{l^\epsilon}_{\hat{A}^\perp, A_c^\perp, B}(1)$ where $(P^{l^\epsilon}_{\hat{A}^\perp, A_c^\perp, B}(u))_{u \in [0,1]}$ is the unique solution of the ODE with values in $\mathrm{Mat}(N, \mathbb{C})$ given by $P^{l^\epsilon}_{\hat{A}^\perp, A_c^\perp, B}(0) = 1_{\mathrm{Mat}(N,\mathbb{C})}$ and

$$\frac{\mathrm{d}}{\mathrm{d}u} P^{l^\epsilon}_{\hat{A}^\perp, A_c^\perp, B}(u) - P^{l^\epsilon}_{\hat{A}^\perp, A_c^\perp, B}(u) \cdot \left(\sum_a T_a(\hat{A}^\perp, f_a^{l^\epsilon}(u)) + (A_c^\perp + B\,\mathrm{d}t)(l'(u)) \right) = 0 \tag{9.2}$$

for all $u \in [0, 1]$. Here "$\cdot$" is the standard multiplication of $\mathrm{Mat}(N, \mathbb{C})$.

We set $WLF(L^\epsilon)(\hat{A}^\perp + A_c^\perp + B\,\mathrm{d}t) := \prod_{i=1}^n \mathrm{Tr}_{\mathrm{Mat}(N,\mathbb{C})}(\mathrm{Hol}(\hat{A}^\perp, l_i^\epsilon; A_c^\perp, B))$. The mapping $\mathcal{N}^* \ni \hat{A}^\perp \mapsto WLF(L^\epsilon)(\hat{A}^\perp + A_c^\perp + B\,\mathrm{d}t) \in \mathbb{C}$ will be denoted by $WLF(L^\epsilon)(\cdot + A_c^\perp + B\,\mathrm{d}t)$.

Proposition 9.1. *For every link L in $\Sigma \times S^1$, every $\epsilon > 0$ and all $A_c^\perp \in \mathcal{A}_c^\perp$, $B \in C^\infty(\Sigma, \mathfrak{t}'_{\mathrm{reg}})$ we have* $\mathrm{WLF}(L^\epsilon)(\cdot + A_c^\perp + B\,\mathrm{d}t) \in (\mathcal{N})$.

Proposition 9.1 is obvious if G is Abelian because then one has $WLF(L^\epsilon)(\cdot + A_c^\perp + B\,\mathrm{d}t) \in \mathcal{E}(\mathcal{N}) \subset (\mathcal{N})$, cf. Eq. (10.2) below. If G is non-Abelian the situation is not so easy anymore but one can obtain a proof for Proposition 9.1 also in the non-Abelian case by adapting the proof of Proposition 6 in [21] in a suitable way.

Remark 9.2. For the proof of Theorem 10.1 below to work it will be necessary to impose a suitable smoothness condition on the function-valued mappings $\Sigma \ni \sigma' \mapsto \psi^\epsilon_{\sigma'} \in C^\infty(\Sigma, \mathbb{R})$, $\epsilon > 0$. Instead of trying to identify an appropriate smoothness condition we will restrict ourselves to special families $(\psi^\epsilon_{\sigma'})_{\epsilon > 0, \sigma' \in \Sigma}$ of the form

$$\psi^\epsilon_{\sigma'} = \frac{1}{N^\epsilon_{\sigma'}} \frac{1}{\epsilon^2} \psi\left(\frac{1}{\epsilon} d_{\mathbf{g}}(\cdot, \sigma')\right) \quad \text{with} \quad N^\epsilon_{\sigma'} := \frac{1}{\epsilon^2} \int \psi\left(\frac{1}{\epsilon} d_{\mathbf{g}}(\sigma, \sigma')\right) \mathrm{d}\mu_{\mathbf{g}}(\sigma)$$

for all $\sigma' \in \Sigma$ and $\epsilon < \epsilon_0$ where ψ is any fixed smooth function $\mathbb{R}_+ \to \mathbb{R}_+$ with $\mathrm{supp}(\psi) \subset [0, 1]$ and the additional property that[4] $\underline{\psi} := \psi \circ \| \cdot \| \in C^\infty(\mathbb{R}^2, \mathbb{R})$ where $\| \cdot \|$ is the standard Euclidean norm on $\mathbb{R}^2$. The last condition implies that $\psi(\frac{1}{\epsilon} d_{\mathbf{g}}(\cdot, \sigma'))$ is indeed C^∞ for each $\sigma' \in \Sigma$ and each sufficiently small $\epsilon > 0$.

9.3. Framing

One could hope that $\lim_{\epsilon \to 0} \Phi_B^\perp(\mathrm{WLF}(L^\epsilon)(\cdot + A_c^\perp + B\,\mathrm{d}t))$ exists for $B \in C^\infty(\Sigma, \mathfrak{t}'_{\mathrm{reg}})$, $A_c^\perp \in \mathcal{A}_c^\perp$, and L contained in a sufficiently large set $\mathcal{L}$ of links in $\Sigma \times S^1$ and that after performing also the $\int \cdots DA_c^\perp$ and $\int \cdots DB$-integrations in Eq. (6.6) one obtains a link

[4] One can show that $\psi \circ \| \cdot \| \in C^\infty(\mathbb{R}^2, \mathbb{R})$ if and only if ψ is the restriction onto $\mathbb{R}_+$ of a smooth symmetric function $\mathbb{R} \to \mathbb{R}$.

invariant. However, when one tries to compute the limit $\lim_{\epsilon\to 0} \Phi^{\perp}_{B}(\mathrm{WLF}(L^{\epsilon})(\cdot + A^{\perp}_{c} + B\,\mathrm{d}t))$ already in the simplest case $G = U(1)$ a problem arises, the so-called "self-linking problem", cf., e.g., [20,21]. Witten suggested in [32] to solve the self-linking problem by using a regularization procedure which he called "framing". We will use an implementation of this framing procedure which is adapted to our "quasi-axial setting" and which works well also for non-Abelian G (cf. again [20,21] for a motivation of this implementation):

We choose a suitable family $(\phi_s)_{s>0}$ of diffeomorphisms of $\Sigma \times S^1$ such that $\phi_s \circ l_k \xrightarrow{s\to 0} l_k, k \leq n$. We can then compute $WLO(L^{\epsilon}, \phi_s; A^{\perp}_{c}, B) := \Phi^{\perp}_{B,\phi_s}(\mathrm{WLF}(L^{\epsilon})(\cdot + A^{\perp}_{c} + B\,\mathrm{d}t))$ where $\Phi^{\perp}_{B,\phi_s}$ is the "deformed version" of $\Phi^{\perp}_{B}$ obtained by deforming the quadratic form $Q^{\perp}_{B}$ on $\mathcal{A}^{\perp} = \mathcal{N}$ in a certain way. Later we let ϵ and s go to zero.

Proposition 9.2. *Let ϕ be a diffeomorphism of $\Sigma \times S^1$ and let $\phi^* : \mathcal{A} \to \mathcal{A}$ denote the pull-back of ϕ. Then the following three statements are equivalent*:

(C1) $\phi^*(\mathcal{A}^{qax}) = \mathcal{A}^{qax}$
(C2) $\phi^*(\mathcal{A}^{qax}(T)) = \mathcal{A}^{qax}(T)$, *where T is any fixed maximal torus of G.*
(C3) *There is a (unique) diffeomorphism $\bar{\phi}$ of Σ, a smooth mapping $v : \Sigma \to S^1$ and a number $\beta \in \{-1, 1\}$ such that $\phi(\sigma, t) = (\bar{\phi}(\sigma), v(\sigma) \cdot t^{\beta})$ holds for all $\sigma \in \Sigma, t \in S^1$, where "$\cdot$" is the standard multiplication of $S^1 \cong U(1)$.*

Proof. It is easy to see that statement (C3) implies both (C1) and (C2). That, conversely, each of the two statements (C1) and (C2) implies (C3) is more difficult to see but not essential for what follows, so we will omit the corresponding proofs. □

Definition 9.2. A diffeomorphism ϕ of $\Sigma \times S^1$ fulfilling the three equivalent conditions (C1)–(C3) will be called "compatible with the quasi-axial gauge".

Let ϕ be a diffeomorphism of $\Sigma \times S^1$ which is compatible with the quasi-axial gauge. One can show that statement (C3) implies $\phi^*(\mathcal{A}^{\perp}) = \mathcal{A}^{\perp}$. In view of this relation it is tempting to try to use the pullback ϕ^* (or rather $(\phi^{-1})^*$) for the deformation of the quadratic form $Q^{\perp}_{B}$ on $\mathcal{A}^{\perp} = \mathcal{N}$. However, it will turn out that this approach will not work, cf. [20,21] for the discussion of a similar question which arises when studying Chern-Simons models on $\mathbb{R}^3$ in axial gauge. Instead the elements of $\mathcal{N} = \mathcal{A}^{\perp}$ ought to transform like vector fields under the deformation by ϕ.

Let $VF(M)$ denote the space of smooth vector fields on M and let us set $VF^{\perp}(M) := \{X \in VF(M) \mid \mathrm{d}t(X) = 0\}$. Let ϕ_* denote the linear automorphism of $VF(M) \otimes \mathfrak{g}$ which is induced by ϕ. ϕ_* does not leave the subspace $VF^{\perp}(M) \otimes \mathfrak{g} \cong \mathcal{A}^{\perp} = \mathcal{N}$ of $VF(M) \otimes \mathfrak{g}$ invariant but $pr \circ \phi_*$ does where pr is the obvious projection $VF(M) \otimes \mathfrak{g} \to VF^{\perp}(M) \otimes \mathfrak{g}$ (here we have used the identification $VF^{\perp}(M) \otimes \mathfrak{g} \cong \mathcal{A}^{\perp}$ which is induced by the standard Riemannian metric on S^1 and the linear mapping $\xi : VF(\Sigma) \to \mathcal{A}_{\Sigma,\mathbb{R}}$ introduced in Section. 9.2). It is not difficult to see that $pr \circ \phi_* : VF^{\perp}(M) \otimes \mathfrak{g} \to VF^{\perp}(M) \otimes \mathfrak{g}$ is a linear automorphism.

The linear automorphism on $\mathcal{N} = \mathcal{A}^{\perp}$ obtained from $pr \circ \phi_*$ by transport with the isomorphism which identifies $VF^{\perp}(M) \otimes \mathfrak{g}$ with $\mathcal{N} = \mathcal{A}^{\perp}$ will be denoted by ϕ_*. A straightforward computation shows that if $j \in \mathcal{A}^{\perp} \cong VF^{\perp}(M) \otimes \mathfrak{g}$ then $\phi_*(j) \in \mathcal{A}^{\perp} \cong VF^{\perp}(M) \otimes \mathfrak{g}$

is given by $(\phi_*(j))(t) = \overline{\phi}_*(j(t))$ for all $t \in S^1$ where $\overline{\phi}_* : T\Sigma \to T\Sigma$ is the ordinary tangent mapping which is induced by $\phi : \Sigma \to \Sigma$ (cf. Proposition 9.2).

Definition 9.3. Let $B \in C^\infty(\Sigma, \mathfrak{t}'_{\text{reg}})$ and let ϕ be a diffeomorphism of $\Sigma \times S^1$ which is compatible with the quasi-axial gauge. By $Q^\perp_{B,\phi}$ we will denote the (continuous) real quadratic form on $\mathcal{N}$ given by $Q^\perp_{B,\phi}(j) = Q^\perp_B(j, \phi_*(j))$ for all $j \in \mathcal{N}$. $Q^\perp_{B,\phi}$ will also denote the real symmetric bilinear form obtained by polarization, i.e., the (continuous) real symmetric bilinear form on $\mathcal{N}$ given by

$$Q^\perp_{B,\phi}(j_1, j_2) = \tfrac{1}{2}\left[Q^\perp_B(j_1, \phi_*(j_2)) + Q^\perp_B(j_2, \phi_*(j_1))\right] \quad \text{for all } j_1, j_2 \in \mathcal{N} \tag{9.3}$$

Definition 9.4. Let B and ϕ be as above. We will denote the Gaussian element of $(\mathcal{N})^*$ corresponding to $(a^\perp_B, -2\pi\lambda i Q^\perp_{B,\phi})$ by $\Phi^\perp_{B,\phi}$ (cf. Section 7).

Definition 9.5. Let $L = (l_1, \ldots, l_n)$, $n \in \mathbb{N}$, be an admissible link in $\Sigma \times S^1$. An admissible framing of L is a family $(\phi_s)_{s>0}$ of diffeomorphisms of $\Sigma \times S^1$ with the following properties:

- (F1) Each ϕ_s, $s > 0$, is compatible with the quasi axial gauge and we have $\phi_s^*(\nu_{\boldsymbol{g}} \wedge \mathrm{d}t) = \nu_{\boldsymbol{g}} \wedge \mathrm{d}t$ where $\nu_{\boldsymbol{g}}$ is the positively oriented volume form associated to $\boldsymbol{g}$.
- (F2) For all $i, j \leq n$ and all sufficiently small $s > 0$ the pair $(l_i, \phi_s \circ l_j)$ is an admissible link in $\Sigma \times S^1$.
- (F3) $\forall\, i \leq n$: $\phi_s \circ l_i \xrightarrow{s\to 0} l_i$ $d_{\boldsymbol{g}}$-uniformly and $\lim_{s\to 0} \mathrm{LK}^*(l_i, \phi_s \circ l_i)$ exists.

Remark 9.3. In [18], we will replace $LK^*(l_i, \phi_s \circ l_i)$ in (F3) by $LK(l_i, \phi_s \circ l_i)$.

10. Computation of the WLOs for the special case $\Sigma = S^2$ and $G = U(1)$

Let us now restrict ourselves to the situation of Eq. (6.4), i.e., let us assume that G is Abelian and $\Sigma = S^2$. Clearly, for Abelian G the bilinear form $Q^\perp_B$ does not depend on B so we can set $Q^\perp := Q^\perp_B$.

In Section 10.1, we will evaluate the inner integral of (6.4) at a rigorous level using the regularization procedures which we have introduced in Section 9. As we will show in [18] it is also possible to give a rigorous meaning to the heuristic (Gauss-type) integral functional $\int \cdots \exp(i(k/2\pi)\langle A^\perp_c, \mathrm{d}B\rangle_\Sigma) DA^\perp_c \otimes DB$. In the present paper, we content ourselves with dealing with this integral functional at a heuristic level. We do this by integrating first against DB and then against $DA^\perp_c$ in Section 10.2 below. This is possible if the link $L = (l_1, \ldots, l_n)$ considered has the additional property that $t_0 \notin \mathrm{Image}(l^j_{S^1})$, $j \leq n$.

10.1. The integration $\int \cdots \mathrm{d}\hat{\mu}^\perp_B$

For simplicity we will only treat the special case $N = 1$ and $G = U(1)$. In this case, we have $T = G = U(1)$ and $\mathfrak{t}'_{\text{reg}} = \mathfrak{t} = u(1)$. The results which we derive for this special Abelian group can be generalized easily to arbitrary $N \in \mathbb{N}$ and arbitrary closed connected Abelian subgroups of $U(N)$. Taking into account Proposition 9.1 we obtain

Theorem 10.1. *Let $L = (l_1, \ldots, l_n)$, $n \in \mathbb{N}$, be an admissible link in $\Sigma \times S^1$ and let $(\phi_s)_{s>0}$ be an admissible framing of L. Then*

$$WLO(L, (\phi_s)_{s>0}; A_c^\perp, B) := \lim_{s \searrow 0} \lim_{\epsilon \searrow 0} \Phi^\perp_{B,\phi_s}(WLF(L^\epsilon)(\cdot + A_c^\perp + B\,\mathrm{d}t)) \quad \textit{exists}$$

for all $A_c^\perp \in \mathcal{A}_c^\perp$, $B \in C^\infty(\Sigma, \mathfrak{t})$ and setting $l_\Sigma^j := (l_j)_\Sigma$, $l_{S^1}^j := (l_j)_{S^1}$, $\mathrm{lk}_j^* := \lim_{s\to 0} \mathrm{LK}^*(l_j, \phi_s \circ l_j)$, and $I_j := (l^j_{S^1})^{-1}(\{t_0\})$ for $j \leq n$, we obtain

$$\begin{aligned} &WLO(L, (\phi_s)_{s>0}; A_c^\perp, B) \\ &\quad = \prod_j \exp(\lambda\pi i\,\mathrm{lk}_j^*) \prod_{j\neq k} \exp(\lambda \pi i\,\mathrm{LK}^*(l_j, l_k)) \\ &\qquad \times \prod_j \left[\exp\left(\int_{l_\Sigma^j} A_c^\perp\right) \exp\left(\sum_{u \in I_j} \mathrm{sgn}(l^j_{S^1}; u) \cdot B(l^j_\Sigma(u))\right)\right] \end{aligned} \tag{10.1}$$

where $\mathrm{sgn}(l^j_{S^1}; u) = 1$ (resp. -1 resp. 0) *if the loop $l^j_{S^1}$ crosses t_0 at u in the direction of the orientation of S^1 (resp. crosses t_0 at u in the opposite direction resp. does not cross but only touches t_0 at u).*

Proof. Let L, n, $(\phi_s)_{s>0}$, $A_c^\perp$, and B be as in the assertion and let $\epsilon \in (0, \epsilon_0)$, where ϵ_0 is as in Section 9.2. Clearly, $i = T_1 \in u(1)$ so setting $f^{l_j^\epsilon}(u) := f_1^{l_j^\epsilon}(u)$ we have

$$\begin{aligned} &WLF(L^\epsilon)(\hat{A}^\perp + A_c^\perp + B\,\mathrm{d}t) \\ &\quad = \exp\left(i\left(\hat{A}^\perp, \sum_k \int_0^1 f^{l_k^\epsilon}(u)\,\mathrm{d}u\right)\right) \prod_j \left[\exp\left(\int_{l^j_\Sigma} A_c^\perp\right) \exp\left(\int_{l_j} B\,\mathrm{d}t\right)\right] \end{aligned} \tag{10.2}$$

so from Definition 9.4 we obtain

$$\begin{aligned} &\Phi^\perp_{B,\phi_s}(WLF(L^\epsilon)(\cdot + A_c^\perp + B\,\mathrm{d}t)) \\ &\quad = \exp\left(i\pi\lambda Q^\perp_{\phi_s}\left(\sum_k \int_0^1 f^{l_k^\epsilon}(u)\,\mathrm{d}u\right)\right) \exp\left(i\, a_B^\perp\left(\sum_j \int_0^1 f^{l_j^\epsilon}(u)\,\mathrm{d}u\right)\right) \\ &\qquad \times \prod_j \left[\exp\left(\int_{l^j_\Sigma} A_c^\perp\right) \exp\left(\int_{l_j} B\,\mathrm{d}t\right)\right] \end{aligned} \tag{10.3}$$

Let us set $l^j_{\mathbb{R}} := i^{-1}_{S^1} \circ l^j_{S^1} - 1/2$, $j \leq n$. Then

$$\lim_{\epsilon \to 0} i a_B^\perp \left(\int_0^1 f^{l_j^\epsilon}(u)\,\mathrm{d}u\right)$$

$$= i \int_0^1 \left[\lim_{\epsilon\to 0} \langle\langle f^{l_j^\epsilon}(u), m(B) \rangle\rangle_{\mathcal{H}} \right] \mathrm{d}u \overset{(*)}{=} \int_0^1 \left[l^j_{\mathbb{R}}(u) \frac{\mathrm{d}}{\mathrm{d}u} B(l^j_\Sigma(u)) \right] \mathrm{d}u \tag{10.4}$$

Here step $(*)$ follows from Eq. (8.7), $\int_{S^1}(i_{S^1}^{-1}(t) - 1/2)\eta^\epsilon_{l^j_{S^1}(u)}(t)\,\mathrm{d}t \xrightarrow{\epsilon\to 0} l^j_{\mathbb{R}}(u)$ for all $u \notin I_j = (l^j_{S^1})^{-1}(\{t_0\})$, and

$$\int_\Sigma \left(T_1 \otimes \xi(PT_{\gamma(l^j_\Sigma(u),\cdot)}((l^j_\Sigma)'(u))\ \psi^\epsilon_{l^j_\Sigma(u)}), \mathrm{d}B \right)_{\boldsymbol{g}} \mathrm{d}\mu_{\boldsymbol{g}}$$
$$= -T_1 \int_\Sigma \psi^\epsilon_{l^j_\Sigma(u)}\ \mathrm{d}B(PT_{\gamma(l^j_\Sigma(u),\cdot)}((l^j_\Sigma)'(u)))\,\mathrm{d}\mu_{\boldsymbol{g}}$$
$$\xrightarrow{\epsilon\to 0} \frac{1}{i}\mathrm{d}B((l^j_\Sigma)'(u))$$
$$= \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}u}B(l^j_\Sigma(u))$$

Here $PT_{\gamma(l^j_\Sigma(u),\cdot)}((l^j_\Sigma)'(u))$ denotes the vector field $\sigma \mapsto PT_{\gamma(l^j_\Sigma(u),\sigma)}((l^j_\Sigma)'(u))$.

On the other hand from $\int_{l_j} B\,\mathrm{d}t = \int_0^1 B(l^j_\Sigma(u)) \cdot (l^j_{S^1})'(u)\,\mathrm{d}u$ and $(l^j_{S^1})'(u) = (l^j_{\mathbb{R}})'(u)$ it follows

$$\exp\left(\int_0^1 l^j_{\mathbb{R}}(u)\frac{\mathrm{d}}{\mathrm{d}u}B(l^j_\Sigma(u))\,\mathrm{d}u\right)\exp\left(\int_{l_j} B\,\mathrm{d}t\right)$$
$$= \exp\left(\int_0^1 \left\{ l^j_{\mathbb{R}}(u)\frac{\mathrm{d}}{\mathrm{d}u}B(l^j_\Sigma(u)) + B(l^j_\Sigma(u)) \cdot (l^j_{\mathbb{R}})'(u) \right\} \mathrm{d}u \right) \tag{10.5}$$

Let $n_j := \# I_j$ and let $(s_i)_{i\le n_j}$ be the unique strictly increasing sequence of $[0,1]$ such that $I_j = \{s_i \mid i \le n_j\}$. Without loss of generality let us assume that $0 \in I_j$. Restricted onto each of the open intervals (s_i, s_{i+1}) the curve $l^j_{\mathbb{R}}$ will be C^1 and for $u \in (s_i, s_{i+1})$ we will have $(\frac{\mathrm{d}}{\mathrm{d}u}B(l^j_\Sigma(u))) \cdot l^j_{\mathbb{R}}(u) + B(l^j_\Sigma(u)) \cdot (l^j_{\mathbb{R}})'(u) = \frac{\mathrm{d}}{\mathrm{d}u}[B(l^j_\Sigma(u)) \cdot l^j_{\mathbb{R}}(u)]$ so we obtain

$$\int_0^1 \left\{ l^j_{\mathbb{R}}(u)\frac{\mathrm{d}}{\mathrm{d}u}B(l^j_\Sigma(u)) + B(l^j_\Sigma(u)) \cdot (l^j_{\mathbb{R}})'(u) \right\} \mathrm{d}u$$
$$= \sum_i \int_{s_i}^{s_{i+1}} \frac{\mathrm{d}}{\mathrm{d}u}\left[l^j_{\mathbb{R}}(u) \cdot B(l^j_\Sigma(u)) \right] \mathrm{d}u = \sum_i \mathrm{sgn}(l^j_{S^1}; s_i) \cdot B(l^j_\Sigma(s_i))$$

because $\mathrm{sgn}(l^j_{S^1}; s_i) = \lim_{s\uparrow s_i} l^j_{\mathbb{R}}(s) - \lim_{s\downarrow s_i} l^j_{\mathbb{R}}(s)$. Thus, we have

$$\lim_{\epsilon\to 0} \exp\left(i a_B^\perp \left(\int_0^1 f^{l_j^\epsilon}(u)\,\mathrm{d}u \right) \right) \exp\left(\int_{l_j} B\,\mathrm{d}t \right) = \prod_i \exp(\mathrm{sgn}(l^j_{S^1}; s_i) \cdot B(l^j_\Sigma(s_i))) \tag{10.6}$$

Finally, we have for $j, k \leq n$, for all $s > 0$ and $\epsilon > 0$

$$
\begin{aligned}
&Q^{\perp}_{\phi_s}\left(\int_0^1 f^{l^{\epsilon}_k}(v)\,\mathrm{d}v,\ \int_0^1 f^{l^{\epsilon}_j}(v)\,\mathrm{d}v\right)\\
&= \int_0^1 \mathrm{d}v \int_0^1 \mathrm{d}u\ Q^{\perp}_{\phi_s}(f^{l^{\epsilon}_k}(v),\ f^{l^{\epsilon}_j}(u))\\
&= \frac{1}{2}\int_0^1 \mathrm{d}v \int_0^1 \mathrm{d}u[Q^{\perp}(f^{l^{\epsilon}_k}(v), (\phi_s)_*(f^{l^{\epsilon}_j}(u))) + Q^{\perp}((\phi_s)_*(f^{l^{\epsilon}_k}(v)),\ f^{l^{\epsilon}_j}(u))]
\end{aligned}
\tag{10.7}
$$

Recall that $(l_j, \phi_s \circ l_k)$ is admissible if $s > 0$ is sufficiently small (cf. (F2) in Definition 9.5). So from Lemma 4 and Lemma 5 below and (F3) in Definition 9.5 we obtain

$$
\begin{aligned}
&\lim_{s\to 0}\lim_{\epsilon\to 0} Q^{\perp}_{\phi_s}\left(\int_0^1 f^{l^{\epsilon}_k}(v)\,\mathrm{d}v,\ \int_0^1 f^{l^{\epsilon}_j}(v)\,\mathrm{d}v\right)\\
&= \frac{1}{2}\lim_{s\to 0}\left[\mathrm{LK}^*(l_k, \phi_s \circ l_j) + \mathrm{LK}^*(\phi_s \circ l_k, l_j)\right] = \begin{cases} \mathrm{LK}^*(l_j, l_k), & \text{if } j \neq k \\ \mathrm{lk}^*_j, & \text{if } j = k \end{cases}
\end{aligned}
\tag{10.8}
$$

Eq. (10.1) now follows from Eqs. (10.3), (10.6), and (10.8). □

Lemma 4. *For every admissible link* $(l, \tilde{l})$ *in* $\Sigma \times S^1$ *we have*

$$
\lim_{\epsilon\to 0} Q^{\perp}\left(\int_0^1 f^{l^{\epsilon}}(v)\,\mathrm{d}v,\ \int_0^1 f^{\tilde{l}^{\epsilon}}(u)\,\mathrm{d}u\right) = \mathrm{LK}^*(l, \tilde{l})
$$

Proof. According to the definition of $\mathrm{LK}^*(l, \tilde{l})$ it is enough to show that for all $v', v'', u', u'' \in [0, 1]$ with $v' < v''$, $u' < u''$ and the additional property that $l_{\Sigma}([v', v'']) \cap \tilde{l}_{\Sigma}([u', u''])$ contains exactly one element p of $D := DP(l, \tilde{l})\backslash(DP(l) \cup DP(\tilde{l}))$ we have

$$
\lim_{\epsilon\to 0}\int_{v'}^{v''} \mathrm{d}v \int_{u'}^{u''} \mathrm{d}u\ Q^{\perp}(f^{l^{\epsilon}}(v),\ f^{\tilde{l}^{\epsilon}}(u)) = \frac{1}{2}\epsilon(p) \tag{10.9}
$$

Let $\bar{v} \in [v', v'']$, $\bar{u} \in [u', u'']$ be given by $p = l_{\Sigma}(\bar{v}) = \tilde{l}_{\Sigma}(\bar{u})$. For simplicity let us assume in the sequel that $\bar{v}, \bar{u} \notin \{0, 1\}$ (it is not difficult to generalize the proof to the general situation). Then it is easy to see that there is a $\delta > 1$ such that for sufficiently small $\epsilon > 0$ we have

$$
\int_{v'}^{v''} \mathrm{d}v \int_{u'}^{u''} \mathrm{d}u\ Q^{\perp}(f^{l^{\epsilon}}(v),\ f^{\tilde{l}^{\epsilon}}(u)) = \int_{\bar{v}-\delta\epsilon}^{\bar{v}+\delta\epsilon} \mathrm{d}v \int_{\bar{u}-\delta\epsilon}^{\bar{u}+\delta\epsilon} \mathrm{d}u\ Q^{\perp}(f^{l^{\epsilon}}(v),\ f^{\tilde{l}^{\epsilon}}(u)) \tag{10.10}
$$

On the other hand

$$
\begin{aligned}
&Q^{\perp}(f^{l^{\epsilon}}(v),\ f^{\tilde{l}^{\epsilon}}(u))\\
&= \langle\langle f^{l^{\epsilon}}(v), (\partial/\partial t)^{-1} \cdot \star^{-1} \cdot f^{\tilde{l}^{\epsilon}}(u)\rangle\rangle_{\mathcal{H}}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{S^1} \mathrm{d}t \int_{\Sigma} (T_1 \otimes \xi(h^{l^\epsilon}(v)(t)), (\partial/\partial t)^{-1} \cdot \star^{-1} \cdot (T_1 \otimes \xi(h^{\tilde{l}^\epsilon}(u)(t))))_{\boldsymbol{g}}\, \mathrm{d}\mu_{\boldsymbol{g}} \\
&\overset{(*)}{=} \int_{S^1} \mathrm{d}t \int_{\Sigma} \nu_{\boldsymbol{g}}((h^{l^\epsilon}(v))(t), (\partial/\partial t)^{-1}(h^{\tilde{l}^\epsilon}(u))(t))\, \mathrm{d}\mu_{\boldsymbol{g}} \\
&= \int_{S^1} \mathrm{d}t\ \eta^\epsilon_{l_{S^1}(v)}(t)((\partial/\partial t)^{-1} \eta^\epsilon_{\tilde{l}_{S^1}(u)})(t) \\
&\quad \times \int_{\Sigma} \nu_{\boldsymbol{g}} \left(PT_{\gamma(l_\Sigma(v),\cdot)}(l'_\Sigma(v)),\ PT_{\gamma(\tilde{l}_\Sigma(u),\cdot)}(\tilde{l}'_\Sigma(u)) \right) \psi^\epsilon_{l_\Sigma(v)} \psi^\epsilon_{\tilde{l}_\Sigma(u)}\, \mathrm{d}\mu_{\boldsymbol{g}}
\end{aligned}
\tag{10.11}
$$

where the last two "$(\partial/\partial t)^{-1}$" denote the operator $(\partial/\partial t)^{-1} : C^\infty(S^1, \mathbb{R}) \to \tilde{C}^\infty(S^1, \mathbb{R})$ defined in the obvious way. Step $(*)$ above holds because $(\partial/\partial t)^{-1}$ commutes with $\star^{-1}$ and because we have $(\xi(j), \star^{-1} \cdot \xi(j'))_{\boldsymbol{g}} = \nu_{\boldsymbol{g}}(j, j')$ for all $j, j' \in VF(\Sigma)$, which is easy to show.

Using (A4) in Definition 9.1 and the real-valued analogue of (8.3) it follows for sufficiently small $\epsilon > 0$

$$
\int_{S^1} \eta^\epsilon_{l_{S^1}(v)}(t)(\partial/\partial t)^{-1} \eta^\epsilon_{\tilde{l}_{S^1}(u)}(t)\, \mathrm{d}t = \frac{1}{2}[1_{\tilde{l}_{S^1}(\bar{u}) < l_{S^1}(\bar{v})} - 1_{l_{S^1}(\bar{v}) < \tilde{l}_{S^1}(\bar{u})}]
\tag{10.12}
$$

where $<$ is the order relation on S^1 which is induced by $i_{S^1} : [0, 1) \to S^1$. Clearly, we have for all $\sigma \in \Sigma$

$$
\nu_{\boldsymbol{g}}(PT_{\gamma(l_\Sigma(v),\sigma)}(l'_\Sigma(v)),\ PT_{\gamma(\tilde{l}_\Sigma(u),\sigma)}(\tilde{l}'_\Sigma(u))) = \nu_{\boldsymbol{g}}(l'_\Sigma(v), \tilde{l}'_\Sigma(u))
\tag{10.13}
$$

So we obtain

$$
\begin{aligned}
&\lim_{\epsilon \to 0} \int_{\bar{v}-\delta\epsilon}^{\bar{v}+\delta\epsilon} \mathrm{d}v \int_{\bar{u}-\delta\epsilon}^{\bar{u}+\delta\epsilon} \mathrm{d}u\ Q^\perp(f^{l^\epsilon}(v), f^{\tilde{l}^\epsilon}(u)) \\
&\overset{(+)}{=} \frac{1}{2}[1_{\tilde{l}_{S^1}(\bar{u}) \le l_{S^1}(\bar{v})} - 1_{l_{S^1}(\bar{v}) \le \tilde{l}_{S^1}(\bar{u})}] \lim_{\epsilon \to 0} \int_{\bar{v}-\delta\epsilon}^{\bar{v}+\delta\epsilon} \mathrm{d}v \int_{\bar{u}-\delta\epsilon}^{\bar{u}+\delta\epsilon} \mathrm{d}u\, \nu_{\boldsymbol{g}}(l'_\Sigma(v), \tilde{l}'_\Sigma(u)) \\
&\quad \times \int_{\Sigma} \psi^\epsilon_{l_\Sigma(v)} \psi^\epsilon_{\tilde{l}_\Sigma(u)}\, \mathrm{d}\mu_{\boldsymbol{g}} \\
&\overset{(*)}{=} \frac{1}{2}[1_{\tilde{l}_{S^1}(\bar{u}) \le l_{S^1}(\bar{v})} - 1_{l_{S^1}(\bar{v}) \le \tilde{l}_{S^1}(\bar{u})}]\ \nu_{\boldsymbol{g}}(l'_\Sigma(\bar{v}), \tilde{l}'_\Sigma(\bar{u})) \\
&\quad \times \lim_{\epsilon \to 0} \int_{\bar{v}-\delta\epsilon}^{\bar{v}+\delta\epsilon} \mathrm{d}v \int_{\bar{u}-\delta\epsilon}^{\bar{u}+\delta\epsilon} \mathrm{d}u \int_{\Sigma} \psi^\epsilon_{l_\Sigma(v)} \psi^\epsilon_{\tilde{l}_\Sigma(u)}\, \mathrm{d}\mu_{\boldsymbol{g}}
\end{aligned}
$$

provided that $\lim_{\epsilon \to 0} \int_{\bar{v}-\delta\epsilon}^{\bar{v}+\delta\epsilon} \mathrm{d}v \int_{\bar{u}-\delta\epsilon}^{\bar{u}+\delta\epsilon} \mathrm{d}u \int_\Sigma \psi^\epsilon_{l_\Sigma(v)} \psi^\epsilon_{\tilde{l}_\Sigma(u)}\, \mathrm{d}\mu_{\boldsymbol{g}}$ exists. Here step $(+)$ follows from (10.11)–(10.13) and step $(*)$ follows because the map $(v, u) \mapsto \nu_{\boldsymbol{g}}(l'_\Sigma(v), \tilde{l}'_\Sigma(u))$ is continuous and does not vanish in the point $(\bar{v}, \bar{u})$ and is therefore either strictly positive or strictly negative on a neighborhood of $(\bar{v}, \bar{u})$. It is not difficult to see that $\epsilon(p)$ is given by

$$
\epsilon(p) = \mathrm{sgn}(\nu_{\boldsymbol{g}}(l'_\Sigma(\bar{v}), \tilde{l}'_\Sigma(\bar{u}))) \cdot [1_{\tilde{l}_{S^1}(\bar{u}) < l_{S^1}(\bar{v})} - 1_{l_{S^1}(\bar{v}) < \tilde{l}_{S^1}(\bar{u})}]
\tag{10.14}
$$

so equation (10.9) will follow once we have shown that

$$\lim_{\epsilon\to 0}\int_{\bar{v}-\delta\epsilon}^{\bar{v}+\delta\epsilon} \mathrm{d}v \int_{\bar{u}-\delta\epsilon}^{\bar{u}+\delta\epsilon} \mathrm{d}u \int_{\Sigma} \psi^{\epsilon}_{l_\Sigma(v)}\psi^{\epsilon}_{\tilde{l}_\Sigma(u)}\,\mathrm{d}\mu_{\boldsymbol{g}} = \frac{1}{|\nu_{\boldsymbol{g}}(l'_\Sigma(\bar{v}),\tilde{l}'_\Sigma(\bar{u}))|} \tag{10.15}$$

We observe that for every chart $x: U \to V \subset \mathbb{R}^2$ of Σ around p such that $\nu_{\boldsymbol{g}}(\partial/\partial x_1, \partial/\partial x_2)(p) = 1$ we have $\nu_{\boldsymbol{g}}(l'_\Sigma(\bar{v}), \tilde{l}'_\Sigma(\bar{u}))) = (l'_\Sigma(\bar{v}))_1(\tilde{l}'_\Sigma(\bar{u}))_2 - (l'_\Sigma(\bar{v}))_2(\tilde{l}'_\Sigma(\bar{u}))_1$ where $(l'_\Sigma(\bar{v}))_i$ and $(\tilde{l}'_\Sigma(\bar{u}))_i$, $i = 1, 2$, denote the coordinates of $l'_\Sigma(\bar{v})$ and $\tilde{l}'_\Sigma(\bar{u})$ w.r.t. to $(\partial/\partial x_1, \partial/\partial x_2)$. If we take this into account and if we also take into account Remark 9.2 above and compare Eq. (10.15) above with equation (22) in [20] we see that Eq. (10.15) holds at least in the special case where the auxiliary Riemannian metric $\boldsymbol{g}$ was chosen such that each $p \in DP(L)$ has an open neighborhood U such that $\boldsymbol{g}$ is Euclidean on U. The proof of (10.15) in the general situation will be given elsewhere. $\square$

Lemma 5. *For every admissible link $(l, \tilde{l})$ in $\Sigma \times S^1$ and every $s > 0$ such that $(\phi_s \circ l, \tilde{l})$ is admissible we have*

$$\lim_{\epsilon\to 0}\int_0^1 \mathrm{d}v \int_0^1 \mathrm{d}u\ Q^{\perp}((\phi_s)_*(f^{l^\epsilon}(v)), f^{\tilde{l}^\epsilon}(u))$$
$$= \lim_{\epsilon\to 0}\int_0^1 \mathrm{d}v \int_0^1 \mathrm{d}u\ Q^{\perp}(f^{(\phi_s\circ l)^\epsilon}(v), f^{\tilde{l}^\epsilon}(u)) \tag{10.16}$$

Lemma 5 can be proved in a similar way as Eq. (29) in [20]. We will give a detailed proof of Lemma 5 elsewhere.

10.2. The integrations $\int \cdots DB$ and $\int \cdots DA_c^{\perp}$

Now that we made rigorous sense of the inner integral in Eq. (6.4) for every fixed admissible framing $(\phi_s)_{s>0}$ we replace $\int \mathrm{WLF}(L)(\hat{A}^{\perp} + A_c^{\perp} + B\,\mathrm{d}t)\mathrm{d}\hat{\mu}_B^{\perp}(\hat{A}^{\perp})$ in Eq. (6.4) by $\mathrm{WLO}(L, (\phi_s)_{s>0}; A_c^{\perp}, B)$ and set

$$\mathrm{WLO}(L, (\phi_s)_{s>0}) := \frac{1}{Z''}\int\int \mathrm{WLO}(L, (\phi_s)_{s>0}; A_c^{\perp}, B)\exp\left(i\frac{k}{2\pi}\langle A_c^{\perp}, \mathrm{d}B\rangle_{\Sigma}\right) DA_c^{\perp} DB$$

For simplicity we will now restrict ourselves to (admissible) links $L = (l_1, \ldots, l_n)$, $n \in \mathbb{N}$, with the additional property $t_0 \notin \mathrm{Image}((l_j)_{S^1})$, i.e., $I_j = \emptyset$, for every $j \leq n$. In this special case, $WLO(L, (\phi_s)_{s>0}; A_c^{\perp}, B)$ does not depend on B and the two integrations $\int \cdots DB$ and $\int \cdots DA_c^{\perp}$ can be performed in a straightforward way. According to (10.1) and Remark 9.1 we then have

$$\mathrm{WLO}(L, (\phi_s)_{s>0}) = \prod_j \exp(\lambda\pi i \mathrm{lk}_j^*) \prod_{j\neq k} \exp(\lambda\pi i \mathrm{LK}^*(l_j, l_k))$$
$$\times \frac{1}{Z''}\int\int \prod_j \exp\left(\int_{l_\Sigma^j} A_c^{\perp}\right)\exp\left(i\frac{k}{2\pi}\langle A_c^{\perp}, \mathrm{d}B\rangle_{\Sigma}\right) DA_c^{\perp} DB$$
$$= \prod_j \exp(\lambda\pi i \mathrm{lk}_j) \prod_{j\neq k} \exp(\lambda\pi i \mathrm{LK}(l_j, l_k)) \tag{10.17}$$

because, informally,

$$\frac{1}{Z''}\int\int\prod_j \exp\left(\int_{l^j_\Sigma} A^\perp_c\right)\exp\left(i\frac{k}{2\pi}\langle A^\perp_c, \mathrm{d}B\rangle_\Sigma\right) DA^\perp_c DB$$

$$\overset{(+)}{=}\frac{1}{Z''}\int\left[\int \exp\left(i\frac{k}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_c\cdot B)\right)DB\right]\prod_j \exp\left(\int_{l^j_\Sigma} A^\perp_c\right)DA^\perp_c$$

$$\overset{(\dagger)}{=}\text{const.}\ \frac{1}{Z''}\int \delta(dA^\perp_c)\prod_j \exp\left(\int_{l^j_\Sigma} A^\perp_c\right)DA^\perp_c$$

$$\overset{(*)}{=}\text{const.}\ \frac{1}{Z''}\int \delta(dA^\perp_c)\exp(0)DA^\perp_c$$

$$=\frac{1}{Z''}\int\int \exp\left(i\frac{k}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_c\cdot B)\right)DBDA^\perp_c \overset{(++)}{=}\frac{Z''}{Z''}=1$$

Step (†) follows from $DB = \mathrm{d}B$ (cf. Subsec. 5.2), step ($*$) follows because $H^1(\Sigma) = 0$ for $\Sigma = S^2$. So $A^\perp_c$ is closed iff it is exact but for exact $A^\perp_c$ one has $\int_{l^j_\Sigma} A^\perp_c = 0$. Steps (+) and (++) follow from Stokes' Theorem.

Remark 10.1. In [18], we will consider the case of general admissible links $L = (l_1, \ldots, l_n)$ in $\Sigma \times S^1$. It will turn out that there is an interesting interplay between the $\mathrm{LK}^*(l_i, l_j)$-, the $\int_{l^j_\Sigma} A^\perp_c$-, and the $\mathrm{sgn}(l^j_{S^1}; u)B(l^j_\Sigma(u))$-expressions in Eq. (10.1) which finally leads to linking number expressions $\mathrm{LK}(l_i, l_j)$ also in the more general situation where the winding numbers of $l^i_{S^1}$ and $l^j_{S^1}$ vanish but I_i and I_j are not necessarily empty[5].

11. Conclusions and Outlook

In the present paper, we studied Chern-Simons models on manifolds of the form $M = \Sigma \times S^1$ using quasi-axial gauge fixing and later also torus gauge fixing. For the case where Σ or the structure group G of the model is simply-connected we exploited the properties of quasi-axial gauge fixing and derived certain heuristic integral expressions for the WLOs of the model. These integral expressions, i.e the right-hand sides of Eq. (6.3) and Eq. (6.4), which is the Abelian special case of (6.3), have some promising features. In particular, the inner integrals in (6.3) and (6.4) are of “Gaussian type”. We expect that because of this it will eventually be possible to find a rigorous realization of these expressions both in the case of Abelian and of non-Abelian G.

Of course, for Abelian G it has already been demonstrated by other methods that it is possible to obtain a rigorous definition of the WLOs in terms of path integral expression, cf., e.g., [3,25,1]. However, when using an approach based on Lorentz gauge fixing like

[5] The winding numbers of $l^i_{S^1}$ and $l^j_{S^1}$ vanish iff l_i and l_j are 0-homologous. If l_i and $l_j$are not 0-homologous then $\mathrm{LK}(l_i, l_j)$ is in general not defined

in [3,25] or one that is based on a suitable discretization of the base manifold like in [1] the difference between the Abelian and the non-Abelian situation seems to be so large that completely new techniques are required for the treatment of the non-Abelian case. Quasi-axial gauge fixing has the big advantage that when this gauge-fixing procedure is applied the difference between the Abelian and the non-Abelian situation becomes much smaller. Not surprisingly, there is also a price we have to pay when we want to use quasi-axial gauge fixing: due to the appearance of some rather singular expressions on the right-hand side of Eq. (6.4) the treatment of the Abelian case is, at least at first look, more difficult than the treatment of the Abelian case in [3,25,1]. Anyhow, as we showed in Section 7–10 for the special case $G = U(1)$ it is still possible to make sense of the right-hand side of Eq. (6.4) by using regularization procedures like “loop smearing” and “framing” and constructions from white noise analysis.

Before one tries to find a rigorous realization of the right-hand side of the non-Abelian generalization Eq. (6.3) of Eq. (6.4) it is reasonable to discuss first the question whether, by using torus gauge fixing instead of quasi-axial gauge fixing, things can be simplified even further. For non-compact Σ torus gauge fixing is a “proper” gauge in the sense that at least every “regular” connection (i.e., every element of $\mathcal{A}_{\mathrm{reg}}$) is gauge equivalent to a connection in the torus gauge, cf. Proposition 3.4 and Remark 3.1. So Eq. (6.6) makes sense for non-compact Σ and we expect that one can evaluate the WLOs explicitly for arbitrary G by using the “cluster decomposition” technique developed in [21] for the study of Chern-Simons models on the non-compact manifold $\mathbb{R}^3$ in axial gauge.

On the other hand, due to certain topological obstructions, torus gauge fixing can not be applied without modifications if Σ is compact. In other words one should expect that Eq. (6.6), to which one is lead if one neglects the topological obstructions just mentioned, has to be replaced by a more complicated equation. The considerations in [10–12] where torus gauge fixing was exploited for the study of the partition functions of Chern-Simons models on manifolds of the form $\Sigma \times S^1$ give already a certain idea of how this modification of Eq. (6.6) has to look like. We will come back to this point in [18]. In the present paper, we have restricted ourselves mainly to the study of the inner integral in (6.6), which will most probably not be affected by the modification of (6.6) just mentioned. This study can thus be seen as the first step of a program which, as we expect, will eventually lead both to a rigorous path integral representation for the WLOs of non-Abelian Chern-Simons models on the compact manifold $M = S^2 \times S^1$ and to a new, and purely geometric derivation of the R-matrices of Jones and Turaev.

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